A NOTE ON REGULAR GROUP RINGS

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The purpose of this note is to prove the following two theorems.

THEOREM 1. *If* R *is the group ring of a group* G *with respect to a ring* A *with identity, and if* R *is regular, then* G *is a torsion group, and* A *is uniquely divisible by the order of each element in* G.

THEOREM 2. *If* G *is a locally finite group and* A *is a regular ring with identity which is uniquely divisible by the order of every element in* G, *then the group ring of* G *with respect to* A *is regular.*

These theorems answer a question raised by I. Kaplansky. They have been proved by M. Auslander by the methods of homological algebra [1]. The proofs given here are quite elementary. Very similar proofs have been found independently by I. N. Herstein (private communication).

To prove Theorem 1, we may suppose that there is an $x \neq 1$ in G, and we set $u = 1 - x$ in R. Since R is regular, an s exists in R with $usu = u$. We put $t = 1 - su$ and observe that, since u has coefficient sum zero, $t \neq 1$. But $ut = u - usu = 0$, and therefore u is a left zero divisor. We may then write

$$(1 - x)\left(a_0 + \sum_{1}^{N} a_k g_k\right) = 0,$$

where the $a_i \in A$, $a_0 \neq 0$ and the g_k are distinct members of G different from 1. This equation may be rewritten as

$$a_0 + \sum_{1}^{N} a_k g_k - a_0 x - \sum_{1}^{N} a_k (x g_k) = 0,$$

and since no $xg_k = x$, it must happen that some g_k is x and the corresponding a_k is a_0. Renaming the g_k, if necessary, we may rewrite the equation as

$$a_0 + \sum_{2}^{N} a_k g_k - a_0 x^2 - \sum_{2}^{N} a_k (x g_k) = 0.$$

Now if $x^2 \neq 1$, we apply the same argument to obtain

$$a_0 + \sum_{3}^{N} a_k g_k - a_0 x^3 - \sum_{3}^{N} a_k (x g_k) = 0.$$

Received October 3, 1957.

This work was done while the author held a Summer Faculty Research Fellowship from the Rackham School of the University of Michigan.