IMPROPER AFFINE HYPERSPHERES OF CONVEX TYPE AND A GENERALIZATION OF A THEOREM BY K. JÖRGENS

Eugenio Calabi

1. INTRODUCTION

The purpose of this article is to derive certain inequalities satisfied by the elliptic solutions of the nonlinear partial differential equation

$$\det\left(\frac{\partial^2 u}{\partial x^i \partial x^j}\right) = 1 \tag{1.1}$$

with n independent variables $(x) = (x^1, \cdots, x^n)$. An elliptic solution of (1.1) is a function $u = f(x) = f(x^1, \cdots, x^n)$ whose Hessian matrix (symmetric matrix of second derivatives) is definite at each point; that is to say, u is locally either a convex or a concave function (the latter can occur only if n is even) of (x): without loss of generality we shall consider only convex solutions.

The motivation for studying (1.1) is that it represents a special but significant case of the more general equation

$$\det\left(\frac{\partial^2 u}{\partial x^i \partial x^j}\right) = \phi(x)\,, \tag{1.1a}$$

where $\phi(x)$ is a given, positive-valued function; this last equation occurs, for instance, as a result of an elementary transformation of the one arising from the Minkowski problem on closed, convex hypersurfaces in Euclidean $(n + 1)$-space. The outstanding question about the regularity of weak solutions of the Minkowski problem is reduced to the question, if the kth partial derivatives of $\phi(x)$ in (1.1a) satisfy a Hölder condition, whether there exists a convex solution u of (1.1a), locally at least, and whether any such solution has all of its $(k + 2)$nd partial derivatives satisfying a Hölder condition.

It has been shown that (1.1a) has at most one convex solution u in a bounded domain D, if the boundary values of u are prescribed; however a necessary condition for the existence of such a solution with arbitrarily given, but smooth, boundary values is that the domain D be strictly convex. Aside from this, in order to establish the existence of a solution of the boundary value problem by the method of continuity, one requires some *a priori* estimates of the bounds for u, for its second partial derivatives, and for their Hölder constants, in terms of the domain D, the function $\phi(x)$, the boundary data, and their Hölder constants.

The main results of this article (wishfully the first of a series), viewed in the context of the more general problem, provide information on the *a priori* bounds, irrespective of the boundary values of u; the dependence of the bounds on the properties of $\phi(x)$ is set aside for the time being, by considering equation (1.1) instead of the more general one (1.1a). Stated briefly, the inequalities established in the present

---

Received October 6, 1957.

This article was written while the author was supported financially by a research grant from the National Science Foundation.