BOUNDED CONTINUOUS FUNCTIONS ON A LOCALLY COMPACT SPACE

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1. INTRODUCTION

Let X be a compact Hausdorff space, and C[X] the collection of all bounded real-valued continuous functions on X. In the two decades since the appearance of Stone's paper [10], this has been an intensively studied mathematical object. Its interest arises in part from its rich structure; under the uniform norm topology, C[X] is a Banach space (see Myers [7]); under pointwise multiplication, it is an algebra (see Hewitt [3]); under the natural partial ordering, it is a lattice (see Kaplansky [4]). In each case, the underlying compact space X has a faithful representation within the structure of C[X], from which it can be recovered, so that C[X] may be regarded as a tool for the study of X.

When X is no longer compact, the simplicity breaks down. Let C*[X] denote the collection of bounded functions in C[X]. If X is completely regular, then there is associated with it a unique compact space βX, the Stone-Čech compactification of X, and C*[X] and C*[βX] are algebraically isomorphic; X is dense in βX, and every $f \in$ C*[X] has a bounded continuous extension to βX. An analogous pattern holds for C[X] (see [3]). In studying C*[X], we are thus again studying the algebra of all (bounded) continuous functions on a compact space. However, βX can have a very complicated structure, even when X is itself relatively nice (for example, when X is the line). When X is locally compact, something can be achieved by considering the subalgebra $C_0[X]$ of functions on X which vanish at infinity; for this is isomorphic with a fixed maximal ideal in C[X^0], where X^0 is the one-point compactification of X.

In this paper, I shall deal with the full algebra C*[X] for a locally compact space X, with a new topology β; this topology was introduced, in an earlier paper [2], for the special case in which X is a group; it was there called the "strict" topology, because of its resemblance to a topology used by Beurling [1]. In Section 3, the strict topology will be defined, and its properties described; in particular, we prove that it is topologically complete. (This is a considerable improvement on [2], where it was only shown that β is sequentially complete.) In Section 4, we show that the dual space of β-continuous linear functionals in C*[X] is precisely $\mathfrak{M}$[X], the space of bounded Radon measures on X. In Section 5, we obtain a Stone-Weierstrass theorem for β-closed subalgebras of C*[X]. For the sake of completeness, we have extended the preliminary treatment to the space C*[X: E] of bounded continuous functions on the locally compact space X which take values in an arbitrary complete locally convex linear space E. In Section 6, we obtain a type of Stone-Weierstrass theorem for β-closed subspaces of C*[X: E]; this result is of particular interest since the object of study is no longer an algebra of continuous functions, but only a linear space.

2. PRELIMINARY DEFINITIONS

Before introducing the strict topology, we give certain basic definitions, and state several more or less well-known results dealing with continuous functions on

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