TWO APPLICATIONS OF CLOSE-TO-CONVEX FUNCTIONS

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1. In this note we shall extend two recent results on univalent functions. One result concerns the univalence, near $z = \infty$, of certain functions considered by L. Tchakaloff [5]; the other result concerns a domain of univalence of the function $\int_0^z e^{-\zeta^2} d\zeta$, considered by V. S. Rogozhin [4]. We shall make use of the close-to-convex functions introduced by W. Kaplan [2] and Umezawa [6].

2. The following result gives slightly more precise information concerning the domain of univalence of certain functions considered by Tchakaloff [5].

THEOREM 1. *Let* $a_1, a_2, \cdots, a_n$ *be distinct points contained in the disc* $|z - z_0| < R$, *and let* $A_1, A_2, \cdots, A_n$ *be positive constants. Then the function*

$$f(z) \equiv \sum_{k=1}^{n} \frac{A_k}{z - a_k} \tag{1}$$

is univalent in a star-shaped neighborhood of $z = \infty$ *that contains the exterior of the circle* $|z - z_0| = R\sqrt{2}$. *Moreover, the function*

$$g(\zeta) \equiv f\left(\frac{R^2}{\zeta - z_0} + z_0\right) \tag{2}$$

is univalent in a convex domain containing the disc $|\zeta - z_0| < R/\sqrt{2}$.

Proof. We adapt Tchakaloff's proof. If ζ_1 and ζ_2 are distinct points, then (1) and (2) yield the relation

$$g(\zeta_2) - g(\zeta_1) = \frac{(\zeta_2 - \zeta_1)}{R^2} \sum_{k=1}^{n} \frac{A_k}{h_k(\zeta_1, \zeta_2)},$$

where

$$h_k(\zeta_1, \zeta_2) \equiv \left(1 - \frac{(\zeta_2 - z_0)(a_k - z_0)}{R^2}\right)\left(1 - \frac{(\zeta_1 - z_0)(a_k - z_0)}{R^2}\right).$$

Therefore

$$|g(\zeta_2) - g(\zeta_1)| \geq \frac{|\zeta_2 - \zeta_1|}{R^2} \sum_{k=1}^{n} A_k \Re \frac{1}{h_k(\zeta_1, \zeta_2)}, \tag{3}$$

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