ITERATED LIMITS

T. H. Hildebrandt

Let Q_1 and Q_2 be directed sets with order relations R_1 and R_2, respectively, and f a function on Q_1Q_2 to the reals. In this situation the following iterated limits theorem is well known (see Moore and Smith [6, p. 116]): *if* $\lim_{q_1} f(q_1q_2)$ *exists for every* q_2 *and* $\lim_{q_2} f(q_1q_2)$ *exists uniformly for* q_1 *on* Q_1, *then the iterated limits* $\lim_{q_1}\lim_{q_2} f(q_1q_2)$, $\lim_{q_2}\lim_{q_1} f(q_1q_2)$ *and the double limit* $\lim_{q_1q_2} f(q_1q_2)$ *all exist and are equal.* For the case where Q_1 and Q_2 are the positive integers in their natural order, that is, f_{mn} is a double sequence, and f_{mn} is monotone nondecreasing in n for each m, the uniformity condition is also necessary; in other words, if $\lim_m \lim_n f_{mn} = \lim_n \lim_m f_{mn}$, then the inner limits are both uniform, and the double limit exists (see Hildebrandt [4, p. 81]). This note gives the following generalization of this result to Moore-Smith directed limits.

THEOREM. *If* $f(q_1q_2)$ *is a real-valued function on* Q_1Q_2 *such that* $f(q_1q_2)$ *is monotone in* q_1 *in the sense that* $q_1'R_1q_1''$ *implies* $f(q_1'q_2) \geq f(q_1''q_2)$ *for every* q_2, *and if* $\lim_{q_1}\lim_{q_2} f(q_1q_2) = \lim_{q_2}\lim_{q_1} f(q_1q_2)$, *all limits being assumed to exist as finite numbers, then the double limit* $\lim_{q_1q_2} f(q_1q_2)$ *exists and is equal to the iterated limits.*

Let $\lim_{q_1} f(q_1q_2) = g(q_2)$ and $\lim_{q_2} f(q_1q_2) = h(q_1)$, and $\lim_{q_2} g(q_2) = \lim_{q_1} h(q_1) = a$. Then, because of the monotoneity of f in q_1, there exists for every $e > 0$ a q_{2e} such that $q_2R_2q_{2e}$ implies the relation

$$f(q_1q_2) \leq g(q_2) \leq a + 2e.$$

On the other hand, select q_{1e}' so that $h(q_{1e}') \geq a - e$, and q_{2e}' so that $q_2R_2q_{2e}$ implies $f(q_{1e}'q_2) \geq h(q_{1e}') - e$. Then, if $q_1R_1q_{1e}'$ and $q_2R_2q_{2e}'$, it follows from the monotoneity of f that

$$f(q_1q_2) \geq f(q_{1e}'q_2) \geq h(q_{1e}') - e \geq a - 2e.$$

Consequently, if q_{2e}'' is chosen so that $q_{2e}''R_2q_{2e}$ and $q_{2e}''R_2q_{2e}'$, we have that $q_1R_1q_{1e}$ and $q_2R_2q_{2e}''$ implies $a - 2e \leq f(q_1q_2) \leq a + 2e$; in other words, the double limit exists and has the desired value.

Since $\lim_{q_2} g(q_2) = a$, it follows further that for every $e > 0$ there exist q_{1e} and q_{2e} such that if $q_1R_1q_{1e}$ and $q_2R_2q_{2e}$, we have $|f(q_1q_2) - g(q_2)| \leq 2e$, a sort of pseudo-uniformity. In case Q_2 is the set of integers in their natural order, there are only finite number of $n \leq n_e$, for which of course $f(q_1n)$ converges to g(n), so that we actually have uniformity as to n. Since Q_1 and Q_2 are interchangeable, here, we have

COROLLARY 1. *Under the hypothesis of the Theorem, if either* Q_1 *or* Q_2 *is the class of positive integers in their natural order, then the convergence of* f(nq) *is uniform as to* n.

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