AN ELEMENTARY PROOF OF A FUNDAMENTAL THEOREM IN THE THEORY OF BANACH ALGEBRAS

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INTRODUCTION

It is well known that the theory of Banach algebras rests on the Mazur-Gelfand theorem, which states that every complex normed division algebra is isomorphic to the complex field itself. This result, which is directly equivalent to the existence of a spectrum for elements of a normed algebra, was announced by Mazur [5] and proved by Gelfand [2], who used a generalization of the Liouville theorem to vector-valued functions. More recently, elementary proofs of the theorem have been given which avoid complex function theory by an ingenious use of roots of unity [4,7,8]. Another result, fundamental to the theory of Banach algebras and also due to Gelfand [2] in the general case, is the so-called "spectral radius formula," which states that if $\sigma(x)$ is the spectrum of an element x in a Banach algebra with norm $||x||$, then

$$\max_{\lambda\epsilon\sigma(x)} |\lambda| = \lim_{n\to\infty} ||x^n||^{1/n}.$$

The quantity on the left is the spectral radius. Since the formula asserts the existence of a complex number in the spectrum of x with absolute value equal to lim $||x^n||^{1/n}$, it can be regarded as a precise statement concerning the existence of a spectrum. The usual proofs of the spectral radius formula also depend heavily on complex function theory. We give below an elementary proof which avoids function theory, and we obtain, incidentally, another elementary proof of the Mazur-Gelfand theorem. It involves roots of unity in a way similar to the proofs mentioned above; but the proof is different and is indeed considerably simpler, in spite of the greater precision of the result.

1. PRELIMINARIES

We assume that $\mathfrak{A}$ is a complex normed algebra. In other words, $\mathfrak{A}$ is an algebra over the complex field, and the vector space of $\mathfrak{A}$ is a normed linear space whose norm $||x||$ satisfies the multiplicative condition $||xy|| \leq ||x|| \, ||y||$. If $\mathfrak{A}$ is complete in the norm topology, then it is a Banach algebra. In order to deal with the case in which there is no identity element, it is convenient to use the "circle operation" $x \circ y = x + y - xy$, which is associative and has zero as an identity element. An element x is called *quasi-regular (quasi singular)* provided it has (does not have) an inverse, relative to the circle operation. This inverse, if it exists, is called the *quasi-inverse* of x and is denoted by x°. The set Q of all quasi-regular elements in $\mathfrak{A}$ is a group under the circle operation. An important and elementary property of normed algebras is that the mapping $x \to x^\circ$ of Q onto itself is continuous. Arens [1] gives a proof of the corresponding result for the regular elements in a normed algebra with an identity. His proof is adapted to the present case and included here

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