SOME INTEGRAL FORMULAS AND THEIR APPLICATIONS

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0. INTRODUCTION

In a previous paper [9] (see also Yano and Bochner [10]), we have proved the integral formula

$$(0.1) \qquad \int_{V_n} [K_{ji} v^j v^i + (\nabla^j v^i)(\nabla_i v_j) - (\nabla_j v^j)(\nabla_i v^i)]\, d\sigma = 0.$$

The formula is valid for any vector field v^h in an n-dimensional compact orientable Riemannian space V_n, where ∇_j is the operator of covariant differentiation with respect to the Christoffel symbols $\{j{}^h{}_i\}$ formed with the fundamental tensor g_{ji} of V_n, where $\nabla^j = g^{ji}\nabla_i$, where K_{ji} is the Ricci tensor $K_{aji}{}^{a}$, where $K_{kji}{}^{h}$ is the curvature tensor, and where $d\sigma$ is the volume element of the space.

Equation (0.1) can be written in the following three forms:

$$(0.2) \quad \int_{V_n} [K_{ji} v^j v^i + (\nabla^j v^i)(\nabla_j v_i) - \frac{1}{2}(\nabla^j v^i - \nabla^i v^j)(\nabla_j v_i - \nabla_i v_j) - (\nabla_j v^j)(\nabla_i v^i)]\, d\sigma = 0,$$

$$(0.3) \quad \int_{V_n} [K_{ji} v^j v^i - (\nabla^j v^i)(\nabla_j v_i) + \frac{1}{2}(\nabla^j v^i + \nabla^i v^j)(\nabla_j v_i + \nabla_i v_j) - (\nabla_j v^j)(\nabla_i v^i)]\, d\sigma = 0,$$

$$(0.4) \quad \int_{V_n} [K_{ji} v^j v^i - (\nabla^j v^i)(\nabla_j v_i) - \frac{n-2}{n}(\nabla_j v^j)(\nabla_i v^i)$$
$$+ \frac{1}{2}(\nabla^j v^i + \nabla^i v^j - \frac{2}{n} g^{ji}\nabla_b v^b)(\nabla_j v_i + \nabla_i v_j - \frac{2}{n} g_{ji}\nabla_a v^a)]\, d\sigma = 0.$$

From these equations, we can easily obtain

THEOREM A (Myers [5], Bochner [1]; see also Yano and Bochner [10]). *If, in a space* V_n, *the form* $K_{ji}v^j v^i$ *is positive definite, then there does not exist a harmonic vector other than the zero vector.*

THEOREM B (Bochner [1]; see also Yano and Bochner [10]). *If, in a space* V_n, *the form* $K_{ji}v^j v^i$ *is negative definite, then there does not exist a Killing vector other than the zero vector.*

THEOREM C. *If, in a space* V_n, *the form* $K_{ji}v^j v^i$ *is negative definite, then there does not exist a conformal Killing vector other than the zero vector.*

On the other hand, applying Green's formula

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