THE SPACE OF LOOPS ON A LIE GROUP

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INTRODUCTION

In this paper we describe the Hopf algebra $H_*(\Omega'K)$,† where K is a connected compact Lie group and $\Omega'K$ is the e-component of the loop space on K. As an application, we compute the stable homotopy groups $\pi_k(U)$ (by a method other than that envisaged in [5]) and the group $\pi_{2n}\{U(n)\}$. These results, presented in Section 8, have recently been used by Kervaire [11], and independently by Milnor [6], to prove that the n-sphere is parallelizable only if $n = 1, 3$ or 7.

We shall need the following known information about $\Omega'K$:

(1.1) *the space* $\Omega'K$ *is a homotopy-commutative Hopf space;*

(1.2) *both* $H_*(\Omega'K; Q)$ *and* $H^*(\Omega'K; Q)$ *are primitively generated polynomial rings.*

((1.1) follows from the fact that K is itself a Hopf space; (1.2) follows from the Serre C-theory and the fact that over Q, the group K looks like a product of odd spheres [16]). From the application of Morse theory [4], [8], it is further known that

(1.3) *the Z-module* $H_q(\Omega'K)$ *is free* $(q = 0, 1, 2, \cdots)$ *and vanishes for odd* q;

(1.4) *one has an (explicit) additive basis of singular cycles for the classes of* $H_*(\Omega'K)$.

Unfortunately, this basis is not closed under Pontryagin multiplication, and therefore it is not directly applicable to our problem. Nevertheless, the construction of (1.4) is the main tool in the proof of Theorem 1. This theorem, in turn, is the main step towards our description.

Our first new result is that the Pontryagin ring $H_*(\Omega'K)$ is always finitely generated. The explicit generators are described in Theorem 1.

A homomorphism $s: R \to K$ of the real numbers R into K, whose kernel contains the group of integers in R, will be called a *circle on* K. With such a circle we associate two spaces: K_s, the centralizer of the image of s, and K^s, the space K/K_s of left cosets. The formula

$$x \to x\, s(t)\, x^{-1}\, s(t)^{-1} \qquad (x \in K;\ 0 \leq t \leq 1)$$

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†In general we use the singular theory, and we use the lower star for homology, the upper star for cohomology. If no coefficients are indicated, the integers Z are to be understood. The rational numbers are denoted by Q, the reals by R.