A THEOREM OF E. HOPF

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In 1948, E. Hopf published [2] a remarkable theorem to the effect that *the total curvature of a closed surface without conjugate points is nonpositive and vanishes only if the surface is flat.* (Here a Riemannian manifold is said to be without conjugate points if no geodesic contains a pair of mutually conjugate points.) Thanks to the Gauss-Bonnet formula, the latter part of this theorem may be paraphrased: a torus without conjugate points is flat. We have been able to modify Hopf's proof to obtain the following result.

THEOREM. *The integral of the scalar curvature (contracted Riemann tensor) of a compact* C^4 *Riemannian manifold without conjugate points is nonpositive, and it vanishes only if the metric is locally euclidean.*

Here, however, the Gauss-Bonnet-Allendoerfer-Chern-Weil-Fenchel formula does not apply, so that whether an n-dimensional torus without conjugate points is flat is still an open question.

1. ORDINARY DIFFERENTIAL EQUATIONS

Consider the real $m \times m$ matrix differential equation in one independent variable,

$$A''(s) + K(s)A(s) = 0, \tag{J}$$

where $K(s)$ is continuous in s and symmetric. (All differentiations, denoted by dashes, and integrations are entry-wise.) Assume that the solution $A(s)$ with $A(0) = 0$ and $A'(0) = I$ (identity matrix) is such that $\det A(s) \neq 0$ for $s \neq 0$. (This corresponds to the nonconjugacy hypothesis.) Then most of the formalism of the one dependent variable case carries over; in particular, the Wronskian of two solutions A and B, $(A')^*B - A^*B'$, is constant (* denotes transpose). Putting $A = B$, we find that $A'A^{-1}$ is symmetric for $s \neq 0$. Setting

$$B_c(s) = A(s)\int_s^c A^{-1}(t)[A^{-1}(t)]^* \, dt,$$

we see that B_c is a solution of (J) for $0 < s < c$ such that

$$B_c(0) = \lim_{s \to 0+} B_c(s) = I$$

and $B_c(c) = 0$. Since the integrand is symmetric and positive definite, and

$$B_c(s) - B_d(s) = A(s)[B_c'(0) = B_d'(0)],$$

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