MONOTONE MAPPINGS OF MANIFOLDS, II

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1. INTRODUCTION

In [5], I showed that if S is an orientable, n-dimensional generalized closed manifold (that is, an n-gcm), and f(S) = S' is an (n - 1)-monotone mapping of S onto an at most n-dimensional nondegenerate Hausdorff space S', then S' is an orientable n-gcm of the same homology type as S. In the algebraic apparatus used in defining concepts such as homology and monotoneity, a fixed field was assumed as coefficient group. However, if monotoneity is defined over the integers (see also [2]), then S' need only be assumed to be finite-dimensional.

It is the purpose of the present paper to treat the noncompact and nonorientable cases. Two new conditions enter here. In the first place, for the noncompact case the mapping f must be assumed to be *proper*; that is, the counter-images $f^{-1}(M)$ of compact sets M must be compact. The necessity for this is seen from the simple example where $S = E^1$, the open real number interval $[x \mid 0 < x < 1]$. Let S' be the subspace of the cartesian plane consisting of the sets

$$A = [(x, y) \mid 0 < x \leq 1/2,\ y = 0], \qquad B = [(x, y) \mid 4x^2 - 4x + 4y^2 - 4y + 1 = 0],$$

and let p = (1/2, 0). Let f(S) = S' be the identity on A, and map the open interval $1/2 < x < 1$ onto B - p homeomorphically. Then f is monotone but not proper, and S' is not a generalized manifold.

In the second place, for the nonorientable case, it must be assumed, for each $x' \in S'$, that there exists in S an orientable submanifold (more precisely, an orientable n-dimensional generalized manifold, that is, an orientable n-gm) containing $f^{-1}(x')$. This is shown, for the case n = 3, by the following example (the assumption is apparently unnecessary if n = 2; for the classical case, see [3; Lemma 2]): Let P^2 be the projective plane, and S^1 the 1-sphere, and let $S = P^2 \times S^1$; let the field of coefficients be the integers mod 3. Then S is a nonorientable 3-manifold. For some $x \in S^1$, let $M = P^2 \times x$. The open sets containing M that are topologically similar to $P^2 \times E^1$ form a complete system of neighborhoods for M; however, they are nonorientable 3-manifolds. Hence M, although it is acyclic, has no neighborhoods that are orientable 3-manifolds (since the existence of such would induce orientations on sufficiently small elements of the complete neighborhood system described above). Note that there exists a mapping f(S) = S', where S' is the quotient space of S' in which the only points of S that coalesce are the points of M, which is 2-monotone, and that S' is not a 3-gm.

2. THE ORIENTABLE CASE

In a previous paper [6], I have shown that if S and S' are locally compact spaces, and f(S) = S' is a proper, n-monotone (n > 0), continuous mapping, then $\mathfrak{H}^n(S) = \mathfrak{H}^n(S')$. A similar argument shows that f induces a homomorphism of

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