ON CURVATURE AND CHARACTERISTIC OF HOMOGENEOUS SPACES

Hans Samelson

1. This note is concerned with two topics: We show, with a simple geometrical proof, that the Riemannian curvature of a homogeneous space is nonnegative (Thm. I); this had been proved earlier by E. Cartan for symmetric spaces [3]. And we give a new proof for the fact that the Euler-Poincaré characteristic of such a space is nonnegative (Thm. II). It is apparently still unknown whether Theorem II can be deduced from Theorem I by way of the generalized Gauss-Bonnet formula.

Let G be a compact connected Lie group, U a closed subgroup of G, and $M = G/U$ the homogeneous space formed by the left cosets of U; let p be the natural projection of G onto M. The space M is a manifold, and it carries a differentiable structure of class C^∞, induced by p (the C^∞-functions on M are identified with the C^∞-functions on G that are constant on the cosets of U). The group G admits Riemannian metrics that are invariant under left and right translations (bi-invariant); let such a metric be chosen. There is then a Riemannian metric in M, induced in a natural way (see below for a description); and this metric is invariant under the customary action of G on M.

THEOREM I. *All values of the sectional Riemannian curvature of* M, *in the induced Riemannian metric, are nonnegative.*

For a 2-section Σ (two-dimensional subspace of the tangent space) at a point x we denote the sectional Riemannian curvature in direction Σ by $K(x, \Sigma)$.

2. We consider first the special case where M is itself a group.

PROPOSITION 2.1. *All values of the sectional Riemannian curvature in a Lie group* G, *under a bi-invariant metric, are nonnegative. The sectional curvature in direction* Σ *at the identity* e *vanishes if and only if* Σ *generates an abelian subgroup, that is, if and only if the one-parameter groups generated by the vectors in* Σ *commute.*

Proof. It is well known that the geodesics (parametrized proportionally to arc length) of G are exactly the 1-parameter groups in G and their cosets.

Because of transitivity, it is sufficient to consider 2-sections at e. Let Σ be such a section, let X, Y be two vectors spanning Σ (we may and shall assume that they are orthogonal to each other), and let $x(t)$, $y(t)$ be two one-parameter groups whose tangent vectors for $t = 0$ are X and Y, respectively. We map the (t_1, t_2)-plane E^2 into the group G by the map ϕ defined by $\phi(t_1, t_2) = x(t_1) \cdot y(t_2)$. This is easily seen to be a regular map; that is, the differential $\dot{\phi}$ is nonsingular throughout. In fact, the image under $\dot{\phi}$ of the horizontal, respectively vertical unit vector at (t_1, t_2) $\left[\text{that is, } \frac{\partial}{\partial t_1}, \text{ respectively } \frac{\partial}{\partial t_2}, \text{ in customary notation}\right]$ is $x(t_1) \cdot X \cdot y(t_2)$, respectively $x(t_1) \cdot Y \cdot y(t_2)$; we have denoted the action of a left or right translation on a vector simply by left or right multiplication. Since translations are isometries, the two image vectors are independent and orthogonal to each other. The ϕ-images of

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