ON THE TOTAL CURVATURE OF IMMERSED MANIFOLDS, II

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Let M^n be a compact differentiable manifold of dimension n, and let

$$x: M^n \to E^{n+N}$$

be a differentiable mapping of M^n into a Euclidean space of dimension $n + N$ with the property that the functional matrix is everywhere of rank n. Then M^n is said to be immersed in E^{n+N}. If x is one-one, it is said to be imbedded in E^{n+N}. To each unit normal vector $\nu(p)$ of an immersed manifold M^n at $p \in M$, we draw through the origin O of E^{n+N} the unit vector parallel to it. This defines a mapping, to be called $\tilde{\nu}$, of the normal sphere bundle B_ν of M^n into the unit hypersphere S_0 about O. In a previous paper [1; this paper will be referred to as TCI], we studied the volume of the image of $\tilde{\nu}$ and called it the *total curvature* of M^n. It will be advantageous to normalize this volume by dividing it by the volume c_{n+N-1} of S_0, c_{n+N-1} being of course an absolute constant. Throughout this paper, we will understand by the total curvature of M^n the normalized one. Then, if $E^{n+N} \subset E^{n+N'}$ $(N < N')$, the total curvature $T(M^n)$ of M^n remains the same, whether M^n is considered as a submanifold of E^{n+N} or of $E^{n+N'}$ (Lemma 1, Section 1). One of the theorems we proved in TCI states that $T(M^n) \geq 2$. We shall show below (Section 1) that the same argument can be used to establish the following more general theorem.

THEOREM 1. *Let* M^n *be a compact differentiable manifold immersed in* E^{n+N}, *and let* β_i $(0 \leq i \leq n)$ *be its* i*th Betti number relative to a coefficient field. Then the total curvature* $T(M^n)$ *of* M^n *satisfies the inequality*

$$T(M^n) \geq \beta(M^n), \tag{1}$$

where $\beta(M^n) = \sum_{i=0}^{n} \beta_i$ *is the sum of the Betti numbers of* M^n.

The right-hand side of (1) depends on the coefficient field. For the real field, the lower bound in (1) cannot always be attained. In fact, we have the following theorem.

THEOREM 2. *If the equality sign holds in* (1) *with the real field as coefficient field, then* M^n *has no torsion.*

For a compact differentiable manifold M^n given abstractly, the total curvature $T(M^n)$ or $T_x(M^n)$ is a function of the immersion $x: M^n \to E^{n+N}$ (N arbitrary). Obviously, the number $q(M^n) = \inf_x T_x(M^n)$ is a global invariant of M^n itself. Theorem 1 says that $q(M^n) \geq \beta(M^n)$. In this connection, there is another invariant $s(M^n)$ of M^n, namely the minimum number of cells in a cell complex covering M^n. Clearly, we have $s(M^n) \geq \beta(M^n)$. If M^2 is a compact orientable surface of genus g, it is easy to see that

$$q(M^2) = s(M^2) = \beta(M^2) = 2 + 2g.$$

Received October 18, 1957.

Work done while S. S. Chern was under a contract with the National Science Foundation.