ON THE TOTAL CURVATURE OF IMMERSED MANIFOLDS, II

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Let $M^n$ be a compact differentiable manifold of dimension $n$, and let

$$x: M^n \to E^{n+N}$$

be a differentiable mapping of $M^n$ into a Euclidean space of dimension $n + N$ with the property that the functional matrix is everywhere of rank $n$. Then $M^n$ is said to be immersed in $E^{n+N}$. If $x$ is one-one, it is said to be imbedded in $E^{n+N}$. To each unit normal vector $\nu(p)$ of an immersed manifold $M^n$ at $p \in M$, we draw through the origin $O$ of $E^{n+N}$ the unit vector parallel to it. This defines a mapping, to be called $\nu$, of the normal sphere bundle $B_\nu$ of $M^n$ into the unit hypersphere $S_0$ about $O$. In a previous paper [1; this paper will be referred to as TCI], we studied the volume of the image of $\nu$ and called it the total curvature of $M^n$. It will be advantageous to normalize this volume by dividing it by the volume $c_{n+N-1}$ of $S_0$, $c_{n+N-1}$ being of course an absolute constant. Throughout this paper, we will understand by the total curvature of $M^n$ the normalized one. Then, if $E^{n+N} \subset E^{n+N'}$ ($N < N'$), the total curvature $T(M^n)$ of $M^n$ remains the same, whether $M^n$ is considered as a submanifold of $E^{n+N}$ or of $E^{n+N'}$ (Lemma 1, Section 1). One of the theorems we proved in TCI states that $T(M^n) \geq 2$. We shall show below (Section 1) that the same argument can be used to establish the following more general theorem.

**THEOREM 1.** Let $M^n$ be a compact differentiable manifold immersed in $E^{n+N}$, and let $\beta_i$ $(0 \leq i \leq n)$ be its $i$th Betti number relative to a coefficient field. Then the total curvature $T(M^n)$ of $M^n$ satisfies the inequality

$$T(M^n) \geq \beta(M^n),$$

where $\beta(M^n) = \sum_{i=0}^n \beta_i$ is the sum of the Betti numbers of $M^n$.

The right-hand side of (1) depends on the coefficient field. For the real field, the lower bound in (1) cannot always be attained. In fact, we have the following theorem.

**THEOREM 2.** If the equality sign holds in (1) with the real field as coefficient field, then $M^n$ has no torsion.

For a compact differentiable manifold $M^n$ given abstractly, the total curvature $T(M^n)$ or $T_x(M^n)$ is a function of the immersion $x: M^n \to E^{n+N}$ ($N$ arbitrary). Obviously, the number $q(M^n) = \inf_x T_x(M^n)$ is a global invariant of $M^n$ itself. Theorem 1 says that $q(M^n) \geq \beta(M^n)$. In this connection, there is another invariant $s(M^n)$ of $M^n$, namely the minimum number of cells in a cell complex covering $M^n$. Clearly, we have $s(M^n) \geq \beta(M^n)$. If $M^2$ is a compact orientable surface of genus $g$, it is easy to see that

$$q(M^2) = s(M^2) = \beta(M^2) = 2 + 2g.$$

Received October 18, 1957.

Work done while S. S. Chern was under a contract with the National Science Foundation.