SOME UNSOLVED PROBLEMS

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On November 16, 1957, Assumption University of Windsor sponsored a symposium for mathematicians from Ontario, Michigan, and Indiana. The symposium gave occasion for an informal lecture in which I discussed various old and new questions on number theory, geometry and analysis. In the following list, I record these problems, with the addition of references and of a few further questions.

A. NUMBER THEORY

1. It is known [35, Vol. 1, Section 58] that $\pi(2x) < 2\pi(x)$ for sufficiently large x. Is it true that

$$\pi(x + y) \leq \pi(x) + \pi(y) \text{ ?} \tag{1}$$

Ungár has verified the inequality for $y \leq 41$. Hardy and Littlewood [29, p. 69] have proved that

$$\pi(x + y) - \pi(x) < cy/\log y . \tag{2}$$

In the same paper, they discuss many interesting conjectures. They put

$$\limsup_{x \to \infty} [\pi(x + y) - \pi(x)] = \rho(y) ,$$

and they conjecture that $\rho(y) > y/\log y$, and that perhaps $\pi(y) - \rho(y) \to \infty$ as $y \to \infty$.

Hardy and Littlewood deduce (2) by Brun's method. A very difficult conjecture, weaker than (1) but much stronger than (2), is that corresponding to each $\varepsilon > 0$ there exists a y_ε such that, for $y > y_\varepsilon$,

$$\pi(x + y) - \pi(y) < (1 + \varepsilon)y/\log y .$$

It has not yet been disproved that $\rho(y) = 1$ for all y. If $\rho(y) > 1$ for some y, then $\liminf (p_{n+r} - p_n) < \infty$.

2. About seventy years ago, Piltz [38] conjectured that, for each $\varepsilon > 0$, $p_{n+1} - p_n = O(n^\varepsilon)$. Cramér conjectured [7, p. 24] that $p_{n+1} - p_n = O((\log n)^2)$. If

$$\limsup (p_{n+1} - p_n)/(\log n)^2 = 1 ,$$

then, for each $\varepsilon > 0$, infinitely many of the intervals $[n, n + (1 - \varepsilon)(\log n)^2]$ contain no primes, but for $n > n_\varepsilon$, there is a prime between n and $n + (1 + \varepsilon)(\log n)^2$. The Riemann hypothesis implies that

$$p_{n+1} - p_n < n^{\varepsilon+1/2}$$

[35, Vol. 1, p. 338]. Thus the old conjecture that there is always a prime between two consecutive squares already goes beyond the Riemann hypothesis.