

SOME SETS OF SUMS AND DIFFERENCES

F. Bagemihl

Denote by \mathbb{R} the set of real numbers. If $X \subseteq \mathbb{R}$ and $Y \subseteq \mathbb{R}$, let

$$s(X, Y) = \{x + y : x \in X, y \in Y\},$$

$$d(X, Y) = \{x - y : x \in X, y \in Y\} \cup \{y - x : x \in X, y \in Y\},$$

and set $s(X, X) = s(X)$, $d(X, X) = d(X)$.

It is known and easy to prove that $s(X, Y) = d(X, Y) = \mathbb{R}$ if $\mathbb{R} - X$ and $\mathbb{R} - Y$ are both of Lebesgue measure zero or both of first Baire category; in particular, $s(X) = d(X) = \mathbb{R}$ if $\mathbb{R} - X$ is of measure zero or of first category (for an exposition of results of this kind, see [2] and [3]). Suppose that $\mathbb{R} - X$ is of measure zero and $\mathbb{R} - Y$ is of first category; we shall show that it is possible to have

$$s(X, Y) = d(X, Y) \neq \mathbb{R}.$$

Let A be the set of algebraic numbers, L the set of Liouville numbers, and S , T , U the sets of S -, T -, U -numbers, respectively, in Mahler's classification of transcendental numbers (see [4]). No two of the sets A , S , T , U have any elements in common, and $L \subset U$.

LEMMA. L is a residual subset of \mathbb{R} .

Proof. For every natural number n , define E_n to be the set of numbers ξ such that, for all integers p and q with $q > 1$,

$$\left| \xi - \frac{p}{q} \right| \geq \frac{1}{q^n}.$$

If I is any open interval, it contains a rational number p/q (where p and q are integers and $q > 1$), and the intersection of I with the interval $(pq^{-1} - q^{-n}, pq^{-1} + q^{-n})$ contains no point of E_n . This means that E_n is nowhere dense, so that if all the rational numbers as well as the elements of $\bigcup_{n=1}^{\infty} E_n$ are removed from \mathbb{R} , a residual set remains. If ξ belongs to this residual set, then ξ is irrational, and, for every natural number n , there are integers p and q with $q > 1$, such that

$$\left| \xi - \frac{p}{q} \right| < \frac{1}{q^n},$$

which implies that ξ is a Liouville number.

Mahler has proved that $\mathbb{R} - S$ is of measure zero, so that, in a metrical sense, most numbers are S -numbers. The lemma shows, however, that, in a topological sense, most numbers are Liouville numbers. If T should prove to be nonempty, it is nevertheless small in comparison with S and U in the sense that it is both of first category and measure zero. According to the results cited in the second