AN EXAMPLE OF A FUNCTION WITH A DISTORTED IMAGE

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The analogy between measure and measurability on the one hand, and category and possession of the Baire property on the other, is well known (see, for example, [3, pp. 49, 63, 225] and [5, p. 26]); one aspect of it, dealing with rectilinear sections of a plane set, will concern us here.

We shall consider exclusively sets of points in the plane P. Denote by X the set of all real numbers, and by R the set of all positive real numbers. Then

$$P = \{(x, y): x \in X, y \in X\}.$$

For every $x_0 \in X$, let $L_{x_0} = \{(x_0, y): y \in X\}$, and, for every $r \in R$, let

$$C_r = \{(x, y): x^2 + y^2 = r^2\}.$$

According to Fubini [1], if $E \subset P$ and E is (plane Lebesgue) measurable, then there exists a subset X_0 of X of (linear) measure zero such that, for every $x \in X - X_0$, the intersection $E \cap L_x$ is a measurable subset of L_x. If E is a subset of P of measure zero, then there exists a subset X_0 of X of measure zero such that, for every $x \in X - X_0$, the intersection $E \cap L_x$ is a subset of L_x of measure zero. According to Kuratowski and Ulam (see [4] or [3, pp. 223, 222]), if E is a subset of P that possesses the Baire property, then there exists a subset X_1 of X of first category such that, for every $x \in X - X_1$, the subset $E \cap L_x$ of L_x possesses the Baire property. If E is a subset of P of first category, then there exists a subset X_1 of X of first category such that, for every $x \in X - X_1$, the subset $E \cap L_x$ of L_x is of first category.

The converses of these results are false. If f(x) is a function of a real variable, the plane set $J(f) = \{(x, y): y = f(x), x \in X\}$ is called the (geometrical) image of the function f. Sierpiński has shown [6] that there exists a single-valued function whose image is not measurable, and Sierpiński and Zalcwasser have given an example (in [2, p. 85]) of a single-valued function whose image is not of first category and therefore [3, p. 229] does not possess the Baire property. Sierpiński has also proved [6] the existence of a nonmeasurable subset of P which intersects every (straight) line in at most two points.

Let $p \in P$ and $E \subset P$. We say that E is measurable at p if there exists a (circular, open) neighborhood N of p such that $E \cap N$ is a measurable subset of N; otherwise, E is said to be nonmeasurable at p. Similarly, E is of first category at p if there exists a neighborhood N of p such that $E \cap N$ is a subset of N of first category; otherwise, E is of second category at p.

THEOREM. *There exists a function* f(x) $(x \in X)$ *possessing the following properties:*

(a) f *and its inverse are single-valued,*

(b) f *maps the set of real numbers onto itself,*

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