ON POWER SERIES, AREA, AND LENGTH

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Let C be the unit circle, and U the open unit disk in the complex plane. Denote by Φ the class of functions that are holomorphic in U and map every disk that is internally tangent to C onto a Riemann configuration of infinite area, and by Λ the class of functions that are holomorphic in U and map every rectilinear segment in U that terminates in a point of C onto a curve of infinite length. Lohwater and Piranian have established [1] the existence of functions in Φ of the form $\Sigma\, a_k z^k$ with $\Sigma |a_k| < \infty$. The first one of the two theorems which we shall prove implies that, in a certain sense, most functions of this form belong to Φ; we are indebted to Karl Zeller for suggesting a demonstration that is more elementary than our original one. No function $\Sigma\, a_k z^k$ with $\Sigma |a_k| < \infty$ belongs to Λ, however, because such a function maps every radius of U onto a curve whose length is not greater than $\Sigma |a_k|$. Our second theorem asserts that, for every $p > 1$, "most" functions of the form $\Sigma\, a_k z^k$ with $\Sigma |a_k|^p < \infty$ belong to Λ.

For $p \geq 1$, denote by $\mathfrak{L}_p$ the Banach space of all complex sequences $\{a_k\}$ for which $\Sigma |a_k|^p < \infty$; $\|\{a_k\}\| = (\Sigma |a_k|^p)^{1/p}$. With the element $\{a_k\}$ of $\mathfrak{L}_p$, associate the function $\Sigma\, a_k z^k$. If, as $j \to \infty$, the elements $\{a_k^{(j)}\} \in \mathfrak{L}_p$ converge to $\{a_k\} \in \mathfrak{L}_p$, then the sequence of functions $\Sigma\, a_k^{(j)} z^k$ converges uniformly to $\Sigma\, a_k z^k$ on every compact subset of U; this well-known fact is used implicitly in proving below that certain sets, E_m and $F_m(n)$, are closed.

A convex region D is called a tangential domain, if it lies in U, the intersection of its closure and C is the point 1, and the only straight line through the point 1 that does not intersect D is the tangent to C. Let Φ_D be the class of functions that are holomorphic in U and, for every real θ, map the region $D_\theta = \{ze^{i\theta}: z \in D\}$ onto a Riemann configuration of infinite area. Piranian and Rudin have proved [2, Theorem 4] that for every tangential domain D there exists a function in Φ_D of the form $\Sigma\, a_k z^k$ with $\{a_k\} \in \mathfrak{L}_1$; let R_D be the set of all elements of $\mathfrak{L}_1$ whose associated functions belong to Φ_D. If the boundary of D has infinite curvature at the point 1, then $\Phi_D \subset \Phi$.

THEOREM 1. *For every tangential domain* D, R_D *is a residual subset of* $\mathfrak{L}_1$.

Proof. For every natural number m, define E_m to be the set of those elements of $\mathfrak{L}_1$ whose associated functions do not map D_θ for every θ onto a Riemann configuration of area greater than m. Since C is compact, E_m is closed.

Suppose that P(z) is a polynomial and $t > 0$. For any θ, let G_n $(n = 2, 3, 4, \cdots)$ be the intersection of D_θ with the annulus $1 - 1/n < |z| < 1 - 1/2n$. Since D_θ is a tangential domain, the area of G_n is $g(n)/n^2$, where $g(n) \to \infty$ as $n \to \infty$; moreover, in G_n the modulus of the derivative of tz^n is greater than tn/e. Consequently, if n is sufficiently large, the function $P(z) + tz^n$ maps G_n, and hence D_θ for every θ, onto a Riemann configuration of area greater than m. Given $\varepsilon > 0$ and $\{a_k\} \in \mathfrak{L}_1$, choose K so large that $\Sigma_{k=K+1}^{\infty} |a_k| < \varepsilon/2$, set $P(z) = \Sigma_{k=0}^{K} a_k z^k$, let $t = \varepsilon/2$, and take n to be greater than K and so large that, if $b_k = a_k$ $(k = 0, 1, \cdots, K)$, $b_k = \varepsilon/2$ for $k = n$, and $b_k = 0$ for all other nonnegative integers k, we have $\{b_k\} \in \mathfrak{L}_1 - E_m$; clearly, $\|\{a_k\} - \{b_k\}\| < \varepsilon$.

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