ON THE NUMERICAL RANGE OF A BOUNDED OPERATOR

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If T is a continuous linear transformation of a Hilbert space $\mathscr{H}$ into itself, its numerical range W(T) is defined as the set of all complex numbers (Tf, f) with $\|f\| = 1$. The most important facts about W(T) are the following.

(i) W(T) is convex [2, 4, 5, 6].

(ii) The closure of W(T) contains the spectrum of T [5].

(iii) If T is normal, the closure of W(T) is the smallest closed convex set containing the spectrum of T [5].

(iv) If T is normal and W(T) is closed, the extreme points of W(T) are eigenvalues [3].

(v) If W(T) reduces to the single point λ, then $T = \lambda I$, where I is the identity.

(vi) If W(T) is a subset of the real axis, T is self-adjoint.

In this note we obtain a precise description of W(T) for the special case where $\mathscr{H}$ is two-dimensional, and our other results follow very easily from this. Our methods also provide an easy and natural proof for (i), as well as alternative proofs for (v) and (vi).

The reduced angle between the two one-dimensional subspaces determined by f and g in $\mathscr{H}$ is that angle θ in the interval $0 \leq \theta \leq \pi/2$ for which

$$\cos \theta = \frac{|(f, g)|}{\|f\| \, \|g\|}.$$

Our point of departure is the following lemma; most of it is established in [6], but with a rather complicated proof.

LEMMA. *If* T *is a linear transformation of a two-dimensional Hilbert space into itself, its numerical range* W(T) *is an ellipse, the foci of which are the eigenvalues of* T. *If the eigenvalues are distinct, then the eccentricity of the ellipse is* $\sin \theta$, *where* θ *is the reduced angle between the eigenvectors; if there is only one eigenvalue* λ, *the ellipse is a circle with center* λ *and diameter* $\|T - \lambda\|$.

Proof. Since (Tf, f) is a continuous function on the compact surface of the unit sphere, it follows that W(T) is compact. We show that the boundary of W(T) is an ellipse of the required type, and that W(T) contains all interior points of that ellipse.

If T has only the eigenvalue λ, we let $S = T - \lambda I$ and note that $S^2 = 0$. If $S = 0$, then $T = \lambda I$, and W(T) contains only the point λ and is a circle with center λ and diameter $\|T - \lambda I\| = 0$. If S is not 0, there exists an orthonormal pair u and v for which $Su = 0$ and $Sv = \rho u$. It is immediate that $\|S\| = |\rho|$ and that W(S) is a circle of diameter $|\rho|$, hence that W(T) is a circle of diameter $\|T - \lambda I\|$ and center at λ.

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