A THEOREM ON TWO-DIMENSIONAL VECTOR SPACES

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In classical projective geometry, homogeneous coordinates for the line are customarily introduced by means of an algorithm. If one wished to give a formal definition, one might begin by observing that projective lines can be manufactured from two-dimensional vector spaces in a natural way; then a system of homogeneous coordinates for a line L in a projective space might possibly be defined as a one-to-one mapping, from L onto a line so constructed, which preserves projectivities.

More specifically, if V is a two-dimensional vector space over a division ring D, let Π_V be the family of all lines of V which pass through the origin; and for any nonzero vector v of V, let [v] be the unique member of Π_V to which v belongs. A map p: $\Pi_V \to \Pi_V$ will be called a *projectivity* if there is some nonsingular linear transformation α: $V \to V$ such that $[v]p = [v\alpha]$ for all $v \in V$. Then, if L is a line in a projective space P, the map h constitutes a *system of homogeneous coordinates* for L provided, for some vector space V over a division ring D, the map h: $L \to \Pi_V$ is one-to-one onto and p: $\Pi_V \to \Pi_V$ is a projectivity if and only if hph^{-1} is a projectivity of L (where projectivities of L are defined, as classically, to be sequences of perspectivities in P).

The question arises whether such a system of homogeneous coordinates is necessarily equivalent to the one given by the classical algorithm. Put algebraically, this question becomes: if V and W are two-dimensional vector spaces over division rings D and E, respectively, and if f: $\Pi_V \to \Pi_W$ is a one-to-one onto map which preserves projectivities, does there exist a semilinear isomorphism from V onto W which induces f? The map f induces a special isomorphism from the projective group of V onto the projective group of W, and a classical result due to Schreier and van der Waerden [5] tells us that if D and E are commutative and contain more than five elements, then *any* isomorphism between these groups yields an isomorphism of D onto E. Once we know that D and E are isomorphic, then Hua's determination of the automorphisms of the two-dimensional projective groups [4] yields the fact that f is indeed induced by a semilinear isomorphism of V onto W.

We shall show, below, that in general the map f induces either an isomorphism or an anti-isomorphism of D onto E and then, again by Hua's result, f is induced either by a semilinear isomorphism of V onto W, or by a semilinear isomorphism of V onto W* (the dual space of W), followed by the canonical map from W* to W.

We emphasize that our isomorphism of the projective group of V onto the projective group of W is a special one; and whether or not an arbitrary isomorphism yields an isomorphism or anti-isomorphism of D onto E remains an open question.

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THEOREM. *Let* V *and* W *be two-dimensional vector spaces over division rings* D *and* E, *respectively, and suppose* f: $\Pi_V \to \Pi_W$ *is one-to-one onto. Suppose further that if* G *and* H *denote the respective projective groups, then the map* f^*: $G \to H$

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