THE POINCARÉ DUALITY IN GENERALIZED MANIFOLDS

Armand Borel

1. INTRODUCTION

The generalized manifolds, that is, topological spaces having the local homology properties of manifolds, have been studied notably by Čech, Lefschetz, Begle [2, 3], and Wilder [9]; the two last-named authors proved, among other results, a Poincaré duality theorem which is also valid in the noncompact case. The main purpose of this paper is to give a simple proof, within the framework of sheaf theory, of such a theorem. The theorem involves Alexander-Spanier cohomology and Alexander-Spanier cohomology with compact carriers (in the sense of [4], not of [6]; see below), and it is proved in Section 3 under a condition more general than Wilder's, not for the sake of generality, but because this simplifies the exposition. Its relationship to the Begle-Wilder theorem is discussed in Section 7; Sections 4 and 5 introduce local Betti numbers and homological local connectedness; Section 6 is devoted to some results of Wilder which pertain to these notions and are of particular interest for generalized manifolds; the latter are discussed in Section 7.

Notation. All spaces considered here are locally compact (and Hausdorff). $\overline{Y}$ is the closure of a subset Y of the space X; L stands for a principal ideal ring. $C^i(X, L)$ or C^i (resp. $C_c^i(X, L)$ or C_c^i) is the L-module of i-dimensional L-valued Alexander-Spanier cochains in X (resp. with compact carriers) (as defined, for example, in [4a, Exposé VI], under the name of Čech-Alexander cochains of the first (resp. second) kind). $C^*(X, L)$ or C^* (resp. $C_c^*(X, L)$ or C_c^*) is the direct sum of the $C^i(X, L)$ (resp. $C_c^i(X, L)$), endowed with the usual boundary operator raising degrees by one; and $H^*(X, L)$ (resp. $H_c^*(X, L)$) is the resulting cohomology group: the Alexander-Spanier cohomology group (resp. with compact carriers) of X, and with coefficients in L. As is well known, $H^*(X, L)$ may be identified with the Čech cohomology based on infinite coverings, and if $X = Y - F$, with Y compact and F closed in Y, then $H_c^*(X, L)$ may be identified with the relative Čech cohomology group of Y mod F.

By f^* we denote the homomorphism of $H^*(Y, L)$ in $H^*(X, L)$ induced by a continuous map $f: X \rightarrow Y$. In case f is the inclusion of a subspace, it will sometimes be convenient to denote by $H^*(X \subset Y, L)$ the image of f^*.

Let U be an *open* subset of X. Then $C_c^*(U, L)$ may be identified with the subgrating of elements in $C_c^*(X, L)$ having carriers in U; and this embedding gives rise to a homomorphism of $H_c^*(U, L)$ in $H_c^*(X, L)$; it will be denoted by j^* or j^*_{UX}, and its image by $H_c^*(U \subset X, L)$. Recall that, given a closed subset F of X, there is an exact cohomology sequence

$$(1) \qquad \cdots \rightarrow H_c^i(F, L) \rightarrow H_c^{i+1}(X - F, L) \xrightarrow{j^*} H_c^{i+1}(X, L) \rightarrow H_c^{i+1}(F, L) \rightarrow \cdots .$$

As far as sheaf theory is concerned, we use the terminology of [4b] and assume it to be known. *Grating* will stand for *carapace*, and S(a) will denote the carrier (support) of an element a belonging to a grating A. Given a locally finite covering (U_i) $(i \in I)$, a *partition of unity* for A, subordinate to (U_i), is a family (r_i) $(i \in I)$ of

Received June 26, 1957.