A SIMPLIFIED PROOF OF THE PAPPUS-LEISENRING THEOREM

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The title refers to the following generalization to n dimensions of the projective theorem of Pappus:

THEOREM. *Let* S *be a commutative projective* n-*space* $(n > 1)$. *In a hyperplane* H_0 *of* S_n, *let* $T = \{t_i\}$ $(i = 0, 1, \cdots, n)$ *be a set of* $n + 1$ *points no proper subset of which are dependent. Let* $A_k^m = A_m^k$ $(k \neq n)$ *be the subspace determined by* $T - t_k - t_m$, *and through each* A_k^m *let there be passed two hyperplanes distinct from* H_0, *to be denoted by* H_k^m *and* H_m^k. *For each* k, *the* n *hyperplanes* H_k^m $(m = 0, 1, \cdots, k - 1, k + 1, \cdots, n)$ *determine a point* p_k. *Also, for each* m, *the* n *hyperplanes* H_k^m $(k = 0, 1, \cdots, m - 1, m + 1, \cdots, n)$ *determine a point* q_m. *If now the* p_k *are dependent, then so are the* q_m, *and the dependence is of the same rank.*

We shall simplify Leisenring's proof [1] by using an auxiliary point which introduces symmetry and thereby shortens the calculations. Since H_k^m contains p_k and q_m, the Grassmann products $G_k^m = (t_0 t_1 \cdots t_{k-1} p_k t_{k-1} \cdots t_{m-1} q_m t_{m+1} \cdots t_n)$ all vanish. (The order of k and m is not important.) By hypothesis also $(t_0 t_1 \cdots t_n) = 0$ and $(p_0 p_1 \cdots p_n) = 0$. We have to show that $(q_0 q_1 \cdots q_n) = 0$. Let w be any point not incident with H_0. We may write, for $i = (0, 1, \cdots, n)$,

$$p_i = \sum_{j=0}^{n} \lambda_{ij} t_j + w, \qquad \lambda_{jj} = 0, \tag{1}$$

$$q_i = \sum_{j=0}^{n} \mu_{ij} t_j + w, \qquad \mu_{jj} = 0, \tag{2}$$

$$0 = \sum_{j=0}^{n} t_j. \tag{3}$$

The last expression depends only on the choice of coordinate vectors for the t's. From the array one sees that the determinants for the p's and q's have the same form; for the q's, it is

$$\begin{vmatrix} 0 & \mu_{01} & \cdots & \mu_{0n} & 1 \\ \mu_{10} & 0 & \cdots & \mu_{1n} & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & \cdots & 1 & 0 \end{vmatrix}.$$

We shall show that in fact $\mu_{ji} = -\lambda_{ij}$; the theorem then follows on inspection of the determinant.

Received July 18, 1956.