# THE NUMBER OF ORIENTED GRAPHS

Frank Harary

This note is a continuation of the paper [2], whose notation and terminology will be used. A graph G is *oriented* if to each line ab in G precisely one of the two orientations $\overrightarrow{ab}$ and $\overrightarrow{ba}$ is assigned. We note that not all directed graphs are oriented graphs; for in the latter each pair of points is joined by at most one directed line, while in the former there may be two such directed lines, one in each direction. Thus an oriented graph is a one-dimensional oriented simplicial complex. Two oriented graphs are *isomorphic* if there is a one-to-one correspondence between their point sets which preserves directed lines. Let $\theta_{pq}$ be the number of (nonisomorphic) oriented graphs with p points and q lines, and let

$$\theta_p(x) = \sum_{q=0}^{p(p-1)/2} \theta_{pq} x^q \tag{1}$$

be the counting polynomial for oriented graphs with p points. Our object is to obtain a formula for $\theta_p(x)$. As in [2], we use the enumeration methods of Pólya [3].

Davis [1] has recently counted several kinds of binary relations on p objects. His results include a formula for asym (p), the number of nonisomorphic asymmetric relations defined on p objects (a *relation* is a set of ordered couples; it is *asymmetric* if it is both irreflexive and antisymmetric). Thus Davis' number asym (p) is, in our notation $\Sigma_{q=0}^{p(p-1)/2} \theta_{pq}$, that is, the total number or oriented graphs with p points; therefore the formula which we shall obtain is a refinement of that of Davis.

In the framework of Pólya's Theorem (see the Hauptsatz of [3] or [2, Section 2]), an oriented graph of p points is a *configuration* whose figures are the p(p - 1)/2 pairs of points. The *content* of a figure is the number of directed lines it contains, and it is thus zero or one. However, the pair of vertices a and b in a figure can occur in two different directed lines $\overrightarrow{ab}$ and $\overrightarrow{ba}$. Therefore the *figure counting series* $\phi(x)$ for oriented graphs is

$$\phi(x) = 1 + 2x, \tag{2}$$

since there is one figure of content zero and two of content one. In order to apply Pólya's Theorem, it remains to find the cycle index of the configuration group $Q_p$. This group is a permutation group of degree p(p - 1)/2; but as an abstract group it is isomorphic with $S_p$, the symmetric group of degree p.

The *cycle index* of $S_p$ is

$$Z(S_p) = \frac{1}{p!} \sum_{(j)} \frac{p!}{1^{j_1} j_1! \, 2^{j_2} j_2! \cdots p^{j_p} j_p!} f_1^{j_1} f_2^{j_2} \cdots f_p^{j_p}, \tag{3}$$

where the letters $f_k$ are variables and where the summation is taken over all partitions $(j_1, j_2, \cdots, j_p)$ of p satisfying

Received April 9, 1957.