CONTRIBUTIONS TO THE THEORY OF CONVEX BODIES

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1. GENERALIZATION OF THE PRINCIPAL THEOREM OF BRUNN AND MINKOWSKI

The Brunn-Minkowski theorem on closed convex bodies in n-dimensional Euclidean space can be extended by introducing a suitably defined logarithmically convex functional $\rho_K(\vec{x})$. In the present paper we give a proof of such an extension which was announced orally by the author [4], some years ago. Similarly to the way in which the Brunn-Minkowski theorem provides a means for deriving the isoperimetric inequality, our extension leads to a more general inequality. The functional $\rho_K(\vec{x})$ shall be a local function, in the interior and on the boundary of a convex body K, which depends not only on the point $\vec{x}$ ($\vec{x}$ denotes a local vector), but also on K. This dependence shall satisfy the following four conditions.

1. Continuity: If $K_1 \to K_2$ and $\vec{x}_1 \to \vec{x}_2$, then

$$\rho_{K_1}(\vec{x}_1) \to \rho_{K_2}(\vec{x}_2).$$

Here the statement $K_1 \to K_2$ means that the distance between two parallel directed support planes of K_1 and K_2 tends to zero for all directions of these planes.

2. Homogeneity: If $\lambda K + \vec{a}$ denotes a body which is obtained by applying to K a similarity transformation λK and a translation characterized by the vector $\vec{a}$, then the relation

$$\rho_{\lambda K + \vec{a}}(\lambda \vec{x} + \vec{a}) = \lambda^m \rho_K(\vec{x})$$

shall hold.

3. Logarithmic convexity: If

$$K_\theta = (1 - \theta)K_1 + \theta K_2 \qquad (0 \leq \theta \leq 1)$$

denotes the linear combination of K_1, K_2 in the Brunn-Minkowski sense, the inequality

$$\log \rho_{K_\theta}\{(1 - \theta)\vec{x}_1 + \theta \vec{x}_2\} \geq (1 - \theta) \log \rho_{K_1}(\vec{x}_1) + \theta \log \rho_{K_2}(\vec{x}_2) \tag{1}$$

shall hold, where $\vec{x}_1$, $\vec{x}_2$ are arbitrarily chosen points of K_1, K_2, respectively.

4. Nonnegativeness:

$$\rho_K(\vec{x}) \geq 0. \tag{2}$$

It is easily shown that (2) implies that $\rho_K(\vec{x})$ can vanish only at a boundary point.

Example 1. Let $a = \rho_K(\vec{x})$ denote the shortest distance of $\vec{x}$ from the boundary of K. It is evident that a satisfies the conditions 1, 2, 4. We shall prove that

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