ON MATRICES OF TRACE ZERO

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In 1937, K. Shoda [1] showed that if M is any n-rowed square matrix with elements in a field $\mathfrak{F}$ of characteristic zero, and M has trace $\tau(M) = 0$, then there exist square matrices A and B with elements in $\mathfrak{F}$ such that $M = AB - BA$. Shoda's proof is not valid for a field $\mathfrak{F}$ of characteristic p. The purpose of this note is to furnish a proof holding for any field $\mathfrak{F}$. We begin by deriving the following lemma.

LEMMA. *Let* $M = (m_{ij})$ *be an* n*-rowed square matrix with elements in* $\mathfrak{F}$ *such that*

$$\tau(M) = \sum_{i=1}^{n} m_{ii} = 0, \qquad \sum_{i=1}^{n-1} m_{i,i+1} = 0, \qquad m_{ij} = 0 \text{ for } j \geq i + 2.$$

Then $M = AB - BA$, *where* A *and* B *are square matrices with elements in* $\mathfrak{F}$ *and* A *is nonsingular.*

For proof, we let $K = (k_{ij})$ be the n-rowed square matrix with $k_{j+1,j} = 1$ for $j = 1, \cdots, n - 1$ and with all other $k_{ij} = 0$. We also let $B = (b_{ij})$ be the matrix with every $b_{i1} = 0$ and $b_{i,i+3} = 0$ for $i = 1, \cdots, n - 3$. Then the first row of KB is zero and the $(i - 1)$st row of B is the ith row of KB. Also, the $(j + 1)$st column of B is the jth column of BK, and the nth column of BK is zero. Then $H = KB - BK = (h_{ij})$, where

$$(1) \qquad h_{i1} = -b_{i2}, \quad h_{12} = -b_{13}, \quad h_{n-1,n} = b_{n-2,n}, \quad h_{nn} = b_{n-1,n}, \quad h_{ij} = 0 \ (j \geq i + 2),$$

and

$$(2) \qquad h_{ij} = b_{i-1,j} - b_{i,j+1} \qquad [i = 2, \cdots, n;\ j = 2, \cdots, \min(n - 1, i + 1)].$$

It should now be clear that $m_{ij} = h_{ij} = 0$ for $j \geq i + 2$. The other entries h_{ij}, in each column of H except the last, contain a term b_{ij} which does not appear in earlier columns or elsewhere in the same column, and the coefficient of this term is ± 1. It follows that the b_{ij} may be selected successively so that H differs from M in at most two elements, and these are the elements h_{nn} and $h_{n-1,n}$. Since

$$\tau(M) = \tau(H) = \tau(KB - BK) = 0,$$

and $m_{ii} = h_{ii}$ for $i = 1, \cdots, n - 1$, it must be clear that we also have $m_{nn} = h_{nn}$. By the form of H we have

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