CONCERNING A PROBLEM OF ALEXANDROFF

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In his well-known paper [1] *On local properties of closed sets*, P. Alexandroff introduced the notion of *r-dimensional condensation* and employed it to establish the invariance of the property of regular (n - r - 1)-accessibility of a closed set in euclidean n-space. In Section 6, where he discussed the difficulties surrounding the attempt to set up local Betti groups, he indicated that these difficulties vanish when the space is either (1) r-lc or (2) devoid of r-dimensional condensation. In the concluding section of his paper he stated the following problem.

PROBLEM VI. *What relations are there between the absence of condensation* (in all dimensions) *and local connectedness* (also in all dimensions)?

So far as I have been able to find, no one has specifically treated this problem although, as pointed out below, it is partially settled as a corollary of certain theorems in my book *Topology of Manifolds* [2], and while the complementary parts of the solution have been known to me for some time, I have never published these (see the Remarks following the statement of Theorem 2 below). Because of the possible importance of these matters in connection with the application of local Betti groups to lc^n spaces, however, it seems desirable to publish them.

1. IMPLICATIONS OF THE lc^n PROPERTY FOR LACK OF CONDENSATION

In [2], the following theorems are proved.

A. *If the locally compact Hausdorff space* S *is* lc^n, *then* $p^r(x) \leq \omega$ *for all* $x \in S$ *and* $r \leq n$; *and if in addition* S *is semi-*(n + 1)*-connected at some* $\bar{x} \in S$, *then* $p^{n+1}(\bar{x}) \leq \omega$. ([2], p. 211, Th. 2.26.)

B. *If* S *is a locally compact Hausdorff space such that* $p^r(x) \leq \omega$ *for some point* x *of* S, *then* S *has no* r*-dimensional condensation at* x. (([2], p. 358, Cor. 1.12.) Although the proof is given only for the case where x is of countable character—a corresponding proof was also given by Alexandroff ([1], p. 18, Cor. I)—there is little difficulty in revising the proof to remove this restriction.)

Combining these two theorems, we have

THEOREM 1. *If the locally compact Hausdorff space* S *is* lc^n, *then* S *has no* n*-dimensional condensation at any point; and if in addition* S *is semi-*(n + 1)*-connected at some point* x, *then* S *has no* (n + 1)*-dimensional condensation at* x.

Remark. That n-lc would be insufficient to ensure lack of n-dimensional condensation is shown, for instance, by the well-known example

$$S = \{(x, y) \mid 0 < x \leq 1,\ y = \sin 1/x\} \cup \{(0, y) \mid -1 \leq y \leq 1\}, \text{ with } n = 1.$$

Also, that lc^n at x alone is not sufficient to ensure lack of n-dimensional condensation at x is shown by the following example: Let

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