ON THE EMBEDDING OF A GENERALIZED REGULAR RING IN A RING WITH IDENTITY

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1. It is well known that every ring may be embedded in a ring with identity, by a process (to be described below) which was apparently first discussed by Dorroh (see [1]). In the earlier of Stone's famous papers on Boolean rings it is shown that, by use of a suitable special case of this process, each Boolean ring may be embedded in a Boolean ring with identity which is minimal in the class of Boolean rings with identity containing the given ring [5, Theorem 1]. Brown and McCoy extended this result to p-rings [1, Corollary 1 to Theorem 5]. (In fact, they showed that the corresponding extension is minimal in the class of rings of characteristic p.)

We consider here the embedding problem for commutative regular rings and their generalizations. In particular, we find under what conditions a commutative semi-simple ring which is regular, m-regular or π-regular may be embedded in a commutative ring with identity of the same type. (In the case of a regular ring, the assumption of semi-simplicity is, of course, superfluous.) The minimality question will not be taken up, however.

2. We shall confine our attention entirely to commutative rings, merely remarking that some of the material which follows could be presented without the requirement of commutativity.

A commutative ring A is said to be: (1) *regular* if, for each $a \in A$, there is an $x \in A$ satisfying $a^2x = a$; (2) m-*regular* if there is a fixed positive integer m such that, for each $a \in A$, there is an $x \in A$ satisfying $a^{2m}x = a^m$ (in particular, a regular ring may be described as 1-regular); (3) π-*regular* if, for each $a \in A$, there is an $x \in A$ and a positive integer n, depending on a, satisfying $a^{2n}x = a^n$. Regular rings were introduced by von Neumann [4]; m-regular and π-regular rings were first discussed by McCoy [2]. Both of these authors restricted their definitions to rings with identity, but this requirement was subsequently dropped. For some basic properties of regular rings, and for the proof that every regular ring is semi-simple, see [3, pp. 147-149]. (Semi-simplicity is always used in the sense of Jacobson, that is, in the sense that the intersection of the prime maximal ideals is zero.)

If S is a commutative ring with identity such that A admits S as a ring of operators, then A may be embedded in the ring (A; S) with identity defined as follows (see [3, pp. 87-88]): Let $(A; S) = \{(a, s): a \in A, s \in S\}$, and define operations in (A; S) by

$$(a, s) + (b, t) = (a + b, s + t), \quad (a, s) \cdot (b, t) = (ab + sb + ta, st).$$

The identity of (A; S) is the element (0, 1). The subset $A_0 = \{(a, s): s = 0\}$ is easily seen to be an ideal of (A; S) which is isomorphic to the given ring A.

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