NOTE ON AN ENUMERATION THEOREM OF DAVIS AND SLEPIAN

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1. INTRODUCTION

In recent papers, Davis [1] and Slepian [6, 7] independently obtained an elegant combinatorial lemma which proved to be extremely useful for solving certain enumeration problems. Although Davis ([1], equation (1), or briefly [1, (1)]), gives the number of nonisomorphic structures of m-adic relations on n elements, and Slepian [7, (2)] gives the number of types of Boolean functions of n variables, it is clear that their methods are combinatorially identical. The lemma was also found useful by Gilbert [2] in enumerating types of periodic sequences.

Applying this lemma, Slepian [7, (3)] and Davis [1, Theorems 2, 3, 4, 5] obtained formulas for various enumeration problems which are all of the following general form: the desired number of configurations is equal to the reciprocal of the order of the appropriate permutation group, multiplied by a sum (taken over the types of permutations in this group) of terms each of which is the product of the number of group elements of the corresponding type by some power of the number of figures which serve as building blocks for the configurations; in each of these powers, the exponent is equal to the number of cycles in the permutation of this type.

The objects of this note are to state the enumeration formula of Davis and Slepian explicitly, and to show that it may readily be derived from Pólya's enumeration theorem, the Hauptsatz of [5]. We also illustrate the use of the Davis-Slepian Theorem by enumerating the types of Boolean functions of two variables, following Slepian [7] and Pólya [4], and by finding the number of (nonisomorphic) linear graphs on four vertices, following Davis [1], Slepian [6], and Harary [3]. We conclude by stating a formula which combines Pólya's theorem with the Davis-Slepian Theorem, and which is applicable to enumeration problems that involve configurations containing figures whose "content" has more than one dimension.

2. TERMINOLOGY

First we develop a precise statement of the Davis-Slepian Theorem. Let Φ be a set of elements, called *figures*. A *configuration of length* s is a sequence of s figures. Let Γ be a permutation group of degree s and order h. We say that two configurations are *Γ-equivalent* if there exists a permutation in Γ which sends one onto the other. Let b be the number of figures (that is, elements) of Φ, and let B be the number of Γ-equivalence classes of configurations (in other words, the number of Γ-inequivalent configurations). A permutation in Γ, written as a product of disjoint cycles, is of *type* $(j) = (j_1, j_2, \cdots, j_s)$ if it has j_k cycles of length k for $k = 1, \cdots, s$. Let $h_{(j)}$ be the number of elements of Γ of type (j). We now have the notation necessary for stating the formula of Davis and Slepian,

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