DISTRIBUTION OF EIGENVALUES OF CERTAIN INTEGRAL OPERATORS

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1. INTRODUCTION

The classical theorem of H. Weyl concerning the asymptotic behavior of the eigenvalues of the Laplacian can be stated (in three-dimensional space, say) as follows.

Consider the integral equation

(1.1)
$$\frac{1}{2\pi}\int_{\Omega} \frac{\phi(\vec{\rho})}{|\vec{\rho} - \vec{r}|} d\vec{\rho} = \lambda \phi(\vec{r}), \quad \vec{r} \in \Omega,$$

where Ω *is a region. Then*

(1.2)
$$1/\lambda_n \sim \left(\frac{3\pi^2}{\sqrt{2}|\Omega|}\right)^{2/3} n^{2/3}, \quad \text{as } n \to \infty,$$

where $|\Omega|$ *denotes the volume of* Ω. In a previous paper [1] a proof of this theorem, based on the theory of Brownian motion (Wiener measure), was sketched.

It is the purpose of this paper to prove an analogous theorem for integral equations of the form

(1.3)
$$\int_{\Omega} \frac{\phi(\vec{\rho})}{|\vec{r} - \vec{\rho}|^{\alpha/2}} d\vec{\rho} = \lambda \phi(\vec{r}) \quad (0 < \alpha \leq 2).$$

Unlike in the case (1.1), there is no equivalent formulation in terms of a differential equation. The method of proof will be illustrated on the one-dimensional case, and to obtain a somewhat more general result we shall consider the integral equation

(1.4)
$$\int_{-a}^{a} \frac{\phi(y)V(y)}{|y - x|^{\alpha}} dy = \lambda \phi(x),$$

where $V(y)$ is a continuous function bounded away from 0, that is,

(1.5)
$$V(y) > m > 0, \quad (-a \leq y \leq a).$$

To avoid complications of a minor nature we shall consider (1.4) only for $\alpha < 1/2$, indicating later how this restriction can be removed. The final result is given by formula (4.3). The proof will be an adaptation of the argument used in §11 of [1].

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