THE POINT SPECTRUM OF WEAKLY ALMOST PERIODIC FUNCTIONS

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1. INTRODUCTION

We adopt the terminology and notation of our first paper [1] on the family \( \mathcal{W} \) of weakly almost periodic functions on a locally compact Abelian group \( G \). With every w.a.p. function \( x(t) \) we associate a formal Fourier series

\[
\sum_{\lambda \in G^*} a(\lambda)(t, \lambda),
\]

where \( a(\lambda) = M_s[x(s)(-s, \lambda)] \). We show that the Fourier series is the Fourier series of an almost periodic (a.p.) function \( x_i(t) \); that is, every w.a.p. function \( x(t) \) admits a unique decomposition \( x = x_1 + x_2 \), where \( x_1 \) is a.p. and \( M(\|x_2\|^2) = 0 \). The set \( \{ \lambda : a(\lambda) \neq 0 \} \) becomes the discrete or discontinuous part of the spectrum \( \sigma(x) \) (see [2]).

The basic ergodic theorem which underlies the mean value theory of w.a.p. functions in [1] now reappears in the guise of a summability theorem.

2. ABSTRACT SUMMABILITY THEORY

In the present context, summability theory rests on the almost periodic properties of the kernel:

LEMMA 1. Let \( x \) be w.a.p., and let \( y \) be a.p. with the properties \( y(t) > 0 \), \( y(-t) = y(t) \), \( M(y) = 1 \). Then \( x \ast y \) lies in \( \overline{O(x)} \), the closed convex hull of the translates of \( x \).

Proof. For every \( t \),

\[
(x \ast y)(t) = M_s[x(s)y(t-s)] = M_s[x(s)y(s-t)] = \lim_{\alpha} T_\alpha(xy_t),
\]

where the \( T_\alpha \) run through the semi-group of finite convex combinations of translation operators \( x(s) \to x_u(s) = x(s-u) \) ordered by multiplication (see [1], p. 225). Since the \( T_\alpha \) are equi-uniformly continuous and the set \( \{y_t\} \) and (hence) the set \( \{x \cdot y_t\} (t \in G) \) are conditionally compact in the norm topology of \( C(G) \), the convergence is uniform in \( t \). It follows that for every \( \varepsilon > 0 \) there exists a finite set \( \{s_n\} \) in \( G \) and a set \( \{a_n\} \) of positive real numbers with \( \Sigma a_n = 1 \) such that simultaneously

\[
|x \ast y(t) - \Sigma a_n x(s-s_n)y(s-s_n-t)| < \varepsilon/2 \quad (s, t \in G),
\]

\[
\Sigma a_n y(-s_n) = b, \quad \text{where} \quad \left| 1 - \frac{1}{b} \right| < \frac{\varepsilon}{2 \|x\| \cdot \|y\|}.
\]

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