SOME CONSTANTS ASSOCIATED WITH THE RIEMANN ZETA-FUNCTION

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The following proposition was stated without proof by Ramanujan [3, p. 134] and Hardy [2]; two proofs have recently been given by Chowla and the author [1].

THEOREM: *The Riemann zeta-function has the representation*

$$\zeta(s) = \frac{1}{s-1} + \sum_{n=0}^{\infty} \frac{(-1)^n \gamma_n}{n!}(s-1)^n,$$

where

$$\gamma_n = \lim_{N \to \infty}\left[\sum_{t=1}^{N} \frac{\log^n t}{t} - \int_1^N \frac{\log^n x}{x}dx\right].$$

The purpose of this paper is to investigate the magnitudes and signs of the constants γ_n.

1. THE SIGNS OF THE CONSTANTS

THEOREM 1. *Infinitely many* γ_n *are positive, and infinitely many are negative.*

From the identity

$$\zeta(s) = 2^s \pi^{s-1} \sin\frac{s\pi}{2}\, \Gamma(1-s)\zeta(1-s)$$

it follows that

(1) $$\zeta(1-2m) = (-1)^m \frac{2(2m-1)!}{(2\pi)^m}\zeta(2m) \quad (m = 1, 2, 3, \cdots).$$

Comparing this with the power series representation of $\zeta(s)$, one obtains the relation

(2) $$\zeta(1-2m) = -\frac{1}{2m} + \sum_{n=0}^{\infty} \frac{(-1)^n \gamma_n}{n!}(-2m)^n = -\frac{1}{2m} + \sum_{n=0}^{\infty}\frac{\gamma_n}{n!}(2m)^n.$$

From (1) it follows that the sign of $\zeta(1-2m)$ is $(-1)^m$, since all other factors are positive. Hence, if m is positive and even, (2) shows that the γ_n can not all be nonpositive. Assume that there exist only a finite number of positive γ_n, and that N is

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