A THEOREM ON SIMPLE CARDINAL ALGEBRAS

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1. INTRODUCTION

A *cardinal algebra,* as defined by Tarski [1], is an algebraic system which is closed under an operation of countable addition and which satisfies certain axioms abstracted from the common properties of such diverse algebraic systems as the algebras of cardinal numbers, sets, relations, and so forth.

In the framework of a cardinal algebra one can define the concept of an ideal; and from this one can build up a fairly extensive representation theory for such algebras. As is to be expected, the major building blocks in such a theory are the so-called *simple cardinal algebras*, that is, the algebras having no nontrivial proper ideals. The most obvious examples of simple cardinal algebras are the algebra of nonnegative real numbers closed by the addition of ∞, and its three subalgebras: the algebra of nonnegative integers, similarly closed; the two-element algebra $\{0, \infty\}$; and the trivial algebra $\{0\}$. Interestingly enough, these are the only *known* examples; however, no proof that there exist no others has been constructed. This paper does not settle the question; but it gives a sufficient (and trivially necessary) condition for a simple cardinal algebra to be one of the four algebras mentioned above. The condition appears particularly natural in the context of the general representation theory.

Before stating and proving our theorem we shall, for convenience, quote a few definitions and results of Tarski [1] which are required in the sequel.

2. DEFINITIONS AND KNOWN RESULTS

A *cardinal algebra* is defined to be an algebraic system which consists of an underlying set A, a binary operation +, and an operation Σ of countably infinite rank

$$\mathfrak{A} = \langle A, +, \Sigma \rangle,$$

and which satisfies seven axioms. The first five axioms merely imply that Σ is a generalization of +, that there exists a zero element 0, and that the operations are unrestrictedly commutative and associative. The last two axioms are more restrictive. Axiom VI states that if the same element can be written as a sum in two different ways, then the summands have a common subdivision. Axiom VII asserts the existence of a certain type of greatest lower bound.

VI. *If* $\sum_{i<\infty} a_i = \sum_{j<\infty} b_j$, *then there exist elements* c_{ij} *in* A *such that* $a_i = \sum_{j<\infty} c_{ij}$ *and* $b_j = \sum_{i<\infty} c_{ij}$, *for each* i, j.

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