A NOTE ON POWER SERIES WHICH DIVERGE EVERYWHERE ON THE UNIT CIRCLE

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It is well known that there exist power series $\Sigma a_n z^n$ with $\lim a_n = 0$ which diverge for all points z on the unit circle. An example of this kind was constructed by N. Lusin (see [3, pp. 69-71]). The coefficients in Lusin's example have arguments that are everywhere dense in the interval $(0, 2\pi)$, and the question arises as to whether this distribution of the coefficients in the complex plane constitutes an intrinsic property of power series of the type mentioned above. This is by no means the case, as is shown in Theorem 1 below, where the coefficients a_n are all real and nonnegative.

In Theorem 2, we apply the so-called 'doubling process', first used by Erdös, Piranian and the author (see [1, § 3]), in order to construct a cosine series $\Sigma a_n \cos n\theta$, with $a_n \geq 0$ and $\lim a_n = 0$, which diverges for all real values of θ.

THEOREM 1. *There exists a power series* $\Sigma_{n=0}^{\infty} a_n z^n$, *with* $a_n \geq 0$ *and* $\lim a_n = 0$, *which diverges at all points of the unit circle.*

Let

(1)
$$F(z) = \sum_{m=1}^{\infty} z^{k_m} S_m(z)/m,$$

where

(2)
$$S_m(z) = 1 + z^m + z^{2m} + \cdots + z^{(m-1)m}.$$

The exponents k_m in (1) are to be positive integers large enough so that the powers of z in the $(m + 1)$st term of the series are higher than those in the mth term.

Let $e^{i\theta}$ be an arbitrary point on the unit circle, and let h/m be one of the infinitely many rational approximations of $\theta/2\pi$ for which $|\theta/2\pi - h/m| < 1/2m^2$. (See, for instance, [4, p. 48, Theorem 14].) If we put $\theta = (2\pi h + \beta)/m$, then $|\beta| < \pi/m$ and, by (2),

$$\left| S_m(e^{i\theta}) \right| = \left| \sum_{\mu=0}^{m-1} e^{i\mu\beta} \right| > 2m/\pi.$$

(For a proof of this inequality see, for instance, [2, p. 530, Lemma B].) Thus

$$\left| (e^{i\theta})^{k_m} S_m(e^{i\theta})/m \right| > 2/\pi$$

for infinitely many m, and the Taylor series of $F(z)$ cannot converge at the point $z = e^{i\theta}$.