SOME REMARKS ON SET THEORY IV

Paul Erdös

1. SOME PROBLEMS OF SIERPIŃSKI. Sierpiński [6], [7, pp. 9-11] proved that the continuum hypothesis is equivalent with the existence of a decomposition of the plane into two sets S_1 and S_2 such that S_1 is intersected by every horizontal line (and S_2 by every vertical line) in at most a denumerable set. We begin with a generalization of this result.

THEOREM 1. *Assume that* $2^{\aleph_0} = \aleph_1$. *Decompose the set of all lines in the plane into two arbitrary disjoint sets* L_1 and L_2. *Then there exists a decomposition of the plane into two sets* S_1 *and* S_2 *such that each line of* L_i *intersects* S_i $(i = 1, 2)$ *in at most a denumerable set.*

This theorem clearly strengthens one part of Sierpiński's result. To prove the theorem, let $\{l_\alpha\}$ $(\alpha < \Omega_1)$ be a well-ordering of the lines in the plane, and let l_1 belong to L_i. We begin the construction of the sets S_1 and S_2 by assigning all points of l_1 to S_{3-i}. Suppose that for $\beta < \alpha$ the points of the lines l_β have been divided between S_1 and S_2, and that l_α belongs to L_i. Then we assign to S_{3-i} all points of l_α which lie on none of the lines l_β $(\beta < \alpha)$. The sets S_1 and S_2, thus defined by transfinite induction, possess the required properties, since each ordinal less than Ω_1 is denumerable.

If we do not appeal to the continuum hypothesis, our proof gives a decomposition of the plane into two sets S_i $(i = 1, 2)$ such that each line of L_i intersects S_i in a set of power less than $2^{\aleph_0}$. (Compare Sierpiński's remark immediately after Theorem 4 on page 6 of [8].)

The other half of Sierpiński's theorem can also be strengthened. To this end, we want to find necessary and sufficient conditions on two disjoint sets of lines L_1 and L_2 such that the existence of a decomposition of the plane into two sets S_1 and S_2, with every line of L_i intersecting S_i $(i = 1, 2)$ in at most a denumerable set, implies the continuum hypothesis. Such conditions may be stated as follows: Both L_1 and L_2 must contain nondenumerably many lines, and one of them, say L_1, must contain $2^{\aleph_0}$ lines; moreover, there must not exist a point p such that all but $\aleph_1$ lines of L_1 and all but $\aleph_0$ lines of L_2 pass through p.

We suppress the proof, since it is somewhat lengthy and contains no ideas which are not involved in Sierpiński's method [6, p. 2], [7, pp. 10, 11]. Just to give a hint to the reader, we remark that in the proof we distinguish two cases: in Case I, if T is any set of power $\aleph_1$, some line of L_1 does not meet T; in Case II, this condition is not satisfied.

Various problems arise in connection with Theorem 1. Sierpiński [9] proved that the continuum hypothesis is equivalent with the following statement: *Three-dimensional space* E^3 *can be decomposed into three sets* S_i $(i = 1, 2, 3)$ *such that each line parallel to one of the axes* OX_i $(i = 1, 2, 3)$ *intersects* S_i *in a finite set.* This suggests several questions:

a) Distribute the lines in E^3 into three arbitrary sets L_i $(i = 1, 2, 3)$. Does there exist a decomposition of E^3 into three sets S_i such that the intersection of each line of L_i with the corresponding set S_i is finite?

Received by the editors May 18, 1954.