THE CONTENT OF A YOUNG DIAGRAM

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1. INTRODUCTION. As is well known, the substitutional analysis of Alfred Young leads from the diagram $[\lambda]$ to an explicit representation theory of S_n. The purpose of this paper is to throw light on the *graph* $G[\lambda]$ of $[\lambda]$, discussed elsewhere [4, Part III; 5], and on the significance in this context of Frobenius' notation for a partition. It will be helpful to exhibit the ideas involved in a highly intuitive form, and this we attempt to do here.

Consider a doubly infinite matrix $G = (g_{ij})$, where

$$g_{ij} = j - i \quad (i, j = -\infty, \cdots, -1, 0, 1, \cdots, +\infty),$$

and imagine a given $[\lambda]$ superimposed upon G so that the (i, j) node of $[\lambda]$ covers g_{ij}. The operators T and S defined in (2.4) correspond to the horizontal and vertical displacement of $[\lambda]$ over G. The *content* $C[\lambda]$ of $[\lambda]$ corresponds to the set of elements of G covered by $[\lambda]$. In (3.9) we obtain a necessary and sufficient condition that a given content should be admissible, i.e. should correspond to a Young diagram $[\lambda]$, and in §4 we show how to construct $[\lambda]$ when its content is given.

In §5 we pass to the modular theory by replacing the g_{ij} by their nonnegative residues modulo q. The operators T and S are now periodic of order q, so far as the content is concerned. This periodicity shows itself in the fixed content of a q-hook under T and S, which corresponds to a complete set of residues modulo q. This leads in §6 to a criterion that a diagram $[\lambda]$ be a q-core in terms of Frobenius' notation, the criterion having already been given in Young's case [6].

The paper concludes with the adaptation of the familiar partition generating function [2] to yield the content $C[\lambda]$ for all $[\lambda]$.

2. THE CONCEPT OF CONTENT. As a tool in the study of Young diagrams we introduce some concepts associated with the lattice of integer points in the plane. For this theory it is customary to alter the usual coordinate system so that the positive direction is downward on the first axis, towards the right on the second axis. To proceed more formally, let I be the set of integers $\{\cdots, -2, -1, 0, 1, 2, \cdots\}$. We give the name *node* to any element (i, j) of the cartesian product $I \times I$. Let $\{x_i \mid i \in I\}$ be a set of commutative indeterminates. We call x_{j-i} the *content of the node* (i,j) and write

$$x_{j-i} = C\{(i, j)\}. \tag{2.1}$$

If $M = \{(i_1, j_1), \cdots, (i_n, j_n)\}$ is a finite subset of $I \times I$ we define the *content* $C(M)$ of M to be the product of the contents of its nodes, that is,

$$C(M) = \prod_{h=1}^{n} C\{(i_h, j_h)\}. \tag{2.2}$$

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