# ON THE NOTION OF BALANCE OF A SIGNED GRAPH

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This note deals with a generalization of linear graphs. The concepts which it introduces, as well as the manner in which it develops these concepts, were suggested by certain problems in social psychology. The sociometric structure of a group of persons is often represented by means of a square matrix M with a 0 diagonal and with 0's and 1's in the remaining positions. Let $M = \|m_{ij}\|$ ; then $m_{ij}$ is 1 if person $P_i$ likes person $P_j$, and is 0 otherwise. Such matrices correspond to irreflexive binary relations, or to directed graphs (digraphs). The motivation for defining signed graphs arose from the fact that psychologists have also employed square matrices with elements -1, 0, and 1 to represent disliking, indifference, and liking respectively. When a matrix of this sort is symmetric, it can be depicted by an ordinary linear graph which is modified by labelling some of its lines as positive, and the others as negative. Such a modified linear graph will be called a signed graph. If the structure matrix is not symmetric, it can still be represented by means of a directed graph rather than an ordinary graph. Some psychological interpretations of the theory of signed graphs will appear elsewhere.

The standard definitions used in the theory of linear graphs may be found in [3] or, with a social scientific bias, in [2]. However, for the sake of completeness, we shall include some definitions here. A (linear) *graph* G is a collection of n points $P_1, P_2, \cdots, P_n$ together with a given subset L of the set of all unordered pairs of distinct points. The pairs of points which occur in the set L will be called the *lines* of the graph. A *path* of G is a collection of lines of the form $A_1A_2, A_2A_3, \cdots, A_{m-1}A_m$, where the points $A_1, A_2, \cdots, A_m$ are distinct. The *cycle* $A_1A_2\cdots A_m$ of G consists of the path described above together with the line $A_mA_1$. Two points A and B of G are *adjacent* if the line AB is in G. A *complete* graph is one in which each point is adjacent to every other point. A graph is *connected* if every pair of distinct points is joined by a path. A *tree* is a connected graph without cycles.

A *signed graph* G, or briefly an s-*graph*, consists of a set E of n points $P_1, P_2, \cdots, P_n$ together with two disjoint subsets $L^+$, $L^-$ of the set of all unordered pairs of distinct points. The elements of the sets $L^+$, $L^-$ are called *positive lines* and *negative lines* respectively. A *positive cycle* of an s-graph is one in which the number of negative lines is even; a *negative cycle* is not positive. Similarly, the sign of a path is the product of the signs of its lines.

An s-graph is in *balance* if all its cycles are positive. We shall develop a characterization of balance for an abritrary s-graph, and in addition we will describe a procedure for enumerating s-graphs. We begin with a structure theorem for complete balanced s-graphs.

THEOREM 1. *A complete* s-*graph* G *is balanced if and only if its point set* E *is partitioned into two disjoint subsets* $E_1$ *and* $E_2$, *one of which may be empty, such that all lines between points of the same subset are positive and all lines between points of the two different subsets are negative.*

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