THE SPECIAL HOMOTOPY ADDITION THEOREM

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The homotopy addition theorem is an elementary result concerning homotopy groups which is used in proving the Hurewicz isomorphism theorem. It was long considered obvious, and only recently has a proof, given by Sze-tsen Hu, found its way into the literature [3]. This proof of the general theorem entails considerable complications of a technical nature, and it would be even more complicated if Hu did not define homotopy groups in an unusual way, using simplices instead of cubes as basic anti-images. The present paper attempts to avoid some of these complications by considering only that special case of the theorem which is actually needed in the proof of Hurewicz's theorem. Further, it follows the present-day trend by retaining the usual definition of homotopy groups but using the cubic rather than the simplicial singular homology.

Eilenberg's proof of the Hurewicz theorem ([1], pp. 439-444) is readily adapted to the cubic theory, but the (implicit) appeals to the homotopy addition theorem remain. The two final corollaries of the present paper apply the special homotopy addition theorem to this situation.

LEMMA 1. *Let* $u \in F^n(X, x)$ *and*

$$v(x_1, \cdots, x_n) = u(x_1, \cdots, x_{i-1}, 1 - x_{i+1}, x_i, x_{i+2}, \cdots, x_n).$$

Then $v \in F^n(X, x)$ *and* $u \simeq v$. (The notation is defined in [5], pp. 72, 73.)

It is obvious that $v \in F^n$. The homotopy is constructed by twisting the face of I_n lying in the x_i, x_{i+1} -plane through an angle of $\pi/2$ as t goes from 0 to 1. Specifically, the homotopy is defined as follows:

$$H(x_1, \cdots, x_n, t) = u(x_1, \cdots, x_{i-1}, f(x_i, x_{i+1}, t), g(x_i, x_{i+1}, t), x_{i+2}, \cdots, x_n),$$

where

$$f(x, y, t) = (Aa + 1)/2,$$
$$g(x, y, t) = (Ab + 1)/2,$$
$$a(x, y, t) = (2x - 1)\cos \pi t/2 - (2y - 1)\sin \pi t/2,$$
$$b(x, y, t) = (2x - 1)\sin \pi t/2 + (2y - 1)\cos \pi t/2,$$
$$A(x, y, t) = \frac{\max(|2x - 1|, |2y - 1|)}{\max(|a|, |b|)}, \text{ unless } x = y = 1/2, \text{ in which case } A = 0.$$

It is a matter of computation to show that H is well-defined and continuous and provides an admissible homotopy between u and v. (The word 'admissible' is used to emphasize that $\dot{I}_n$ must remain at x during the homotopy.) A is discontinuous at $x = y = 1/2$ (where $|a| = |b| = 0$); but since A is bounded (indeed, $A \leq \sqrt{2}$), f and g are continuous there.

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