A THEOREM ABOUT MAPPINGS OF A TOPOLOGICAL GROUP INTO THE CIRCLE

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1. INTRODUCTION. If S is the unit circle topological group with complex multiplication as its operation, and if G is a compact topological space, then the collection S^G of mappings of G into S with the compact-open topology is a commutative, metric topological group. For f_1 and f_2 in S^G, the product $f_1 \cdot f_2$ is the mapping defined by the relation

$$[f_1 \cdot f_2](x) = f_1(x)f_2(x), \textit{for all } x \in G,$$

and the inverse f^{-1} of any $f \in S^G$ is defined by the relation

$$[f^{-1}](x) = [f(x)]^{-1}, \textit{for all } x \in G.$$

The identity is the mapping f_0 defined by $f_0(x) \equiv 1$. A metric ρ for S is that defined by $\rho(z_1, z_2) = |z_1 - z_2|$, and the metric ρ^* for S^G may be defined by

$$\rho^*(f, g) = \sup_{x \in G} \rho[f(x), g(x)].$$

Following Eilenberg [1], a mapping $f \in S^G$ is said to be equivalent to 1 on a subset H of G ($f \sim 1$ on H) provided there exists a mapping $\phi : H \to R$, the real line, such that $f(x) = \exp[i\,\phi(x)]$ for all $x \in H$. Two mappings f and g in S^G are said to be equivalent on $H \subset G$ ($f \sim g$ on H) provided $f \cdot g^{-1} \sim 1$ on H. It is shown in [1] that $\sim$ is an equivalence relation, that if G is a separable metric space, then $f \sim g$ if and only if f is homotopic to g, and that $P(G) = \{f \mid f \in S^G, f \sim 1\}$ is algebraically a subgroup of S^G. In [1] and here also, if $|z_1 - z_2| < 2$, $[z_1, z_2]$ denotes the signed angle less than π through which the radius to z_1 must be rotated in order to coincide with the radius to z_2.

It will be shown in this paper that $P(G)$ is an open subgroup of S^G when G is compact, and that the factor group $S^G/P(G)$ is isomorphic to the character group of G when G is a compact, connected, commutative topological group satisfying the second axiom of countability. A corollary to this result is the fact that every mapping of such a topological group G into S is homotopic to an interior mapping. This corollary is analogous to a result of G. T. Whyburn [2].

2. With reference to the remark which follows, observe that the function $[z_1, z_2]$, for z_1 and z_2 in S and $|z_1 - z_2| < 2$, is continuous and that $\exp(i[z_1, z_2]) = z_2/z_1$.

(2.1) *If* G *is a compact topological space,* $P(G)$ *is an open subgroup of* S^G.

Proof. Suppose, for some $f \in P(G)$, that g is a mapping such that $\rho^*(f,g) < 2$. Then, for each $x \in G$, $[f(x), g(x)] < \pi$. Since $f \sim 1$ on G, there exists a continuous $\phi : G \to R$ such that $f(x) = \exp[i\phi(x)]$, for all $x \in G$. Let $\psi : G \to R$ be the function defined by

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