SOME REMARKS ON SET THEORY III

by
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This paper continues the treatment of some problems which were discussed in two unnumbered earlier communications [1], [2] of the same title.

1. ON A PROBLEM OF TURÁN. Suppose that with every real number x there is associated a set $S(x)$ of real numbers, called the *picture* of x, and subject to the restriction that x is not contained in $S(x)$. A pair of points x and y is then called *independent* provided neither point is contained in the picture of the other; a set E is called independent if each pair of points in E is independent.

In an oral communication, P. Turán asked whether the hypothesis that each of the pictures $S(x)$ is finite implies the existence of an infinite independent set. Grünwald [5] showed that the answer is in the affirmative. Lázár [7] proved that there exists an independent set of power c. Fodor [3], [4] pointed out that Lázár's proof gives a stronger result: *if no point x is a limit point of its picture $S(x)$, there exists an independent set E with $\overline{\overline{E}} = c$* (throughout this paper, the symbol $\overline{\overline{E}}$ denotes the cardinal number of E).

The remainder of this section deals with other hypotheses on the pictures $S(x)$, and with the question whether or not these hypotheses imply the existence of independent sets of certain cardinalities.

THEOREM 1. *The hypothesis $\overline{\overline{S(x)}} < c$ does not imply the existence of an independent pair.*

Let Ω_c be the least ordinal of power c. We arrange the real numbers in a transfinite sequence

$$\{x_0, x_1, \ldots, x_\xi, \ldots\} (\xi < \Omega_c),$$

and for $x = x_\alpha$ we define

$$S(x) = \{x_\beta \mid \beta < \alpha\}.$$

The example proves the theorem.

Received by the editors in November, 1953.
Research supported in part by the Office of Naval Research.