ON THE STRUCTURE OF RECURRENCE RELATIONS

by

Imanuel Marx

Recurrence relations have long been known for the characteristic solutions of many linear second-order differential equations obtained from the hypergeometric equation. These relations may be found by an examination of the Taylor series for the solutions. Recently a factorization method for deriving characteristic solutions of linear second-order differential equations was reported by L. Infeld and T. E. Hull [1], and their approach includes a partial determination of the coefficients in recurrence relations for the solutions. The equations which may be investigated in this way include the hypergeometric and confluent hypergeometric equations in many forms, as well as a variety of the equations of wave mechanics which, however, are chiefly also of hypergeometric type. Differential equations of the second order with four regular singular points or their confluences present greater difficulty. Of their solutions, only the Mathieu functions, studied by E. T. Whittaker [2], and the radial prolate spheroidal functions, to which Whittaker's method was applied by the writer [3], are known to have recurrence relations. The determination of the coefficients in these relations is complicated by the fact that, for non-hypergeometric equations, neither the characteristic values nor the coefficients in the Taylor series for the characteristic solutions are known functions of the indices.

It is the purpose of this note to present a general framework which appears to include all known recurrence relations, and which may be of assistance in determining coefficients for recurrence relations not yet completely known. The method of Infeld and Hull appears as the simplest case, while the method of Whittaker is seen to be a special example of a very general approach that will yield information about all equations of interest.

The differential equation to be discussed is

$$d/dZ[P(Z)\, dY(Z)/dZ] + [R(Z) + t\,S(Z)]\, Y(Z) = 0. \tag{1}$$

It is assumed that the functions P, R, and S satisfy suitable conditions over a given range of the variable Z, and that a sequence t_n of characteristic values of the parameter t, and a corresponding sequence of characteristic solutions $Y_n(Z)$ of (1), are determined by additional conditions. The functions Y_n are so normalized that a single function $Y(Z, t)$, analytic in t for each fixed Z which is not a singular value of the differential equation, coincides with Y_n for $t = t_n$ and with Y_{n-1} for $t = t_{n-1}$.

By a well-known transformation of the independent and dependent variables in (1), the equation can be replaced by another in which $P \equiv 1$. Other

Received by the editors January 8, 1954.

A portion of the work reported here was done at the Willow Run Research Center of the University of Michigan.