A THEOREM IN PROJECTIVE N-SPACE EQUIVALENT TO COMMUTATIVITY

by

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In any geometry satisfying the rudimentary projective incidence axioms an algebra of points can be introduced on any line, with operations defined by incidences, and by this means an "intrinsic" coordinate system can be introduced. If in this algebra of points, multiplication is commutative, we say the geometry is commutative. It is well known that the Pappus theorem, as an incidence relation on lines in the 2-planes of the space, is a necessary and sufficient condition for the commutativity of the geometry The question arises whether in dimensions above 2 one can state such a condition in terms of the proper elements of the geometry--points and hyperplanes.

For 3-space it has been shown by Reidemeister and Schönhardt (1) that the existence of "Möbius configuration" is such a condition--this being a pair of tetrahedra each of which circumscribes the other. Schönhardt showed by projection that such a configuration implies the existence of the Pappus configuration. The theorem of this paper is a generalization of both the Pappus theorem and a theorem in 3-space (theorem A) equivalent to the existence of the Möbius configuration.

Theorem A. Let T be a plane quadrangle in a commutative projective 3-space, and let a distinct plane be passed through each of the six sides of T. The vertices of T fall into four triangles; let the planes be grouped correspondingly to determine four points--these four points are coplanar.

It is not difficult to see how this theorem is related to the Möbius configuration, but it is susceptible of a simple direct proof, by the Grassman calculus, which is of interest in itself. (This calculus assumes commutativity.)

It can be readily verified that if the six planes are a, b, c, x, y, z so arranged that the vertices of T are the four points [xyz], [xbc], [ayc], [abz] (outer products) then the points required to be dependent are [abc], [ayz], [xbz], [xyc], obtainable from the preceding by simply interchanging a and x, b and y, c and z. By hypothesis the outer product of the vertices of T vanishes; upon expansion this gives the scalar equation

$$[xyza][aycb][cbxz] = [abcx][xbzy][zyac].$$

But one sees that the interchange which would make this equation express conclusion, in fact merely interchanges the two sides.