INVOLUTION AND EQUIVALENCE

by

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The attitude taken in what follows may be briefly characterized by saying that we want to see how far we can go in mathematics using only a minimal logical apparatus. We want to avoid using the concepts of set, element, existence, uniqueness, identity, and, if possible, negation. We do use the concept of relation and implication. In this sense one may discuss the minimal content of a mathematical theory. The procedure is to take a conventional mathematical theory, to express it in terms of the properties of a relation and then to consider only those properties that can be formulated in terms of implication.

This attitude is the same as that taken in a paper on ternary relations in the first volume of this Journal (pp. 97 to 111). The simplest example of a minimal theory in the above sense considered there is the customary definition of a equivalence relation. It consists of three propositions (p. 109):

1. (aa)'
2. (ab)' implies (ba)'
3. (ac)', (bc)' imply (ab)'

In this paper (ab)' is to be read: a and b are equivalent. We want to consider the minimal content of the concept of <u>involution</u> or involutory transformation, or of periodic transformation of period two. The conventional theory from which we start deals with a set S of elements x, y, z etc. To every element x is assigned uniquely an element y=f(x) and if y is assigned to x then x is assigned to y.

The first step consists in introducing instead of the operation y=f(x) a binary relation which we shall denote by (xy)*; at this stage we will say that (xy)* means that y is assigned to x: and since, as we said above, in this case x is assigned to y we can say that (xy)* implies (yx)*; in other words, the relation is symmetric. If it were only that we wanted to translate the original theory in terms of relations we would have to say that given x there exists a y such that (xy)*; and furthermore, that there is only one such y. The first part of this statement has no counterpart in our theory; the discussion of the second part we begin by reformulating it (still within the conventional theory) by saying that if there are two elements y and z each of which is in involution with x - they must be identical; in other words, that (xy)* and (xz)* imply that y is the same as z, or identical with z. But we do not want to use the concept of identity in the ultimate theory, so we replace the statement at this stage by a milder statement; namely, we'll say that if (xy)* and (xz)* then y and z are identical <u>relative to the re-</u>