REMARKS ON A PREVIOUS PAPER[1]

by

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1. Raoul Bott suggested that a proof of Theorems 1 and 2 of [3] which avoided metrical considerations might show the duality relation of Theorem 2 as a consequence of standard duality theorems for convex cones. This method is followed here[2]; it will be seen to shorten the proofs almost to nothing.

It is convenient to take Theorem 1 in a slightly generalized form: Let the cones $U \subset E^k$ and $V \subseteq E^n$ satisfy $D\{U\} = E^k$ and $(D\{V^+\}) = E^n$. Then $AU = V \cap D\{AU\}$ implies $A'V^+ = U^+ \cap D\{A'V^+\}$.

Proof. First, $AU = V \cap D\{AU\}$ gives $(AU)^+ = V^+ + (D\{AU\})^+$; therefore $A'(AU)^+ = A'V^+ + A'(D\{AU\})^+$. Now for any $W \subseteq E^k$ one has $A'(AW)^+ = W^+ \cap A'E^n$. Also in the present case $D\{AU\} = AD\{U\} = AE^k$ and $D\{A'V^+\} = A'D\{V^+\} = A'E^n$. Therefore $A'(AU)^+ = U^+ \cap D\{A'V^+\}$ and $A'V^+ + A'(D\{AU\})^+ = A'V^+ + E^{k+} \cap A'E^n = A'V^+$. The statement has been proved.

Furthermore, $A'V^+$ is affine-equivalent to the geometric polar of AU.

Proof: The latter is $(AU)^+ \bmod (D\{AU\})^+ = (AU)^+ \bmod (AE^k)^+$. Now A' can be considered as defined on $E^n \bmod (AE^k)^+$, since $(AE^k)^+$ is its null-space; so considered, it is one-one, and an affine isomorphism.

Thms. 1 and 2 follow from the above by setting $U = P^k = U^+$, $V = P^n = V^+$; indeed, $D\{P^k\} = E^k$, $D\{P^n\} = E^n$, as required.

2. The following simple construction seems, surprisingly enough, to be new.

Theorem 3. Every pointed convex polyhedral cone is affine-equivalent to the intersection of the positive orthant (in space of appropriate dimension) with a linear subspace.

Proof. Let the cone be $AP^k \subseteq E^n$ (where A may be chosen so that extreme rays of P^k go into extreme rays of AP^k). Consider $A'E^n \cap P^k$. This cone may be represented in the form BP^m (with extreme rays of P^m going into extreme rays of BP^m). Now $BP^m = P^k \cap D\{BP^m\}$, so $B'P^k$

[1] See [3]. The present note follows the terminology and notation of [1] and [3]. Numbers in brackets refer to the Bibliography.

[2] The positive polar of a cone may be regarded as a cone in the dual space, in which case all the proofs in this paper are of affine character. But the dual space is not distinguished notationally.