NORMED FIELDS OVER THE REAL AND COMPLEX FIELDS [1]

by Leonard Tornheim

A normed field F over a field K, where K is contained in the complex field, is a field containing K and for which there is a real valued function $\|y\|$ called a norm satisfying the following conditions:

$$(1) \qquad \|y\| > 0 \text{ if } y \neq 0,$$

$$(2) \qquad \|y + z\| \leqq \|y\| + \|z\|,$$

$$(3) \qquad \|yz\| \leqq \|y\| \cdot \|z\|,$$

$$(4) \qquad \|cy\| = |c| \, \|y\|,$$

where y, z are in F and c is in K.

We shall prove the following results:

THEOREM. <u>Every normed field over the real field</u> R <u>is either the real field</u> R <u>or the complex field</u> C.

COROLLARY. <u>The complex field is the only normed field over the complex field</u>.

These results are not new. Closely related theorems have been stated or proved by Mazur, Gelfand, Arens, Kaplansky, and Ramaswami (see bibliography). But all of their proofs use complex variable theory. Here, however, no use is made of the theory of functions of a complex variable nor of the completion of F. Ostrowski has proved the weaker theorem which has equality in (3).

LEMMA 1. <u>If</u> $\|y\| \leqq 1/2$, <u>then</u> $\|1/(1-y)\| \leqq 2\|1\|$ <u>wherever</u> $1/(1-y)$ <u>exists.</u>

For, $1/(1-y) = 1 + y/(1-y)$. Hence

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