D. K. KAZARINOFF'S INEQUALITY FOR TETRAHEDRA

Nicholas D. Kazarinoff

1. INTRODUCTION

Let S be a tetrahedron, and P a point not exterior to S. Let the distances from P to the vertices and to the faces of S be denoted R_i and r_i, respectively. In this paper we establish an analogue of the Erdös-Mordell inequality for triangles [3, p. 12].

THEOREM 1. *For any tetrahedron whose circumcenter is not an exterior point,*

$$\sum R_i / \sum r_i > 2\sqrt{2}, \tag{1}$$

and $2\sqrt{2}$ *is the greatest lower bound.*

D. K. Kazarinoff stated that this inequality holds for all tetrahedra [3, p. 120]; but he refused to divulge his proof, probably because it was not simple enough, in his opinion, to be made public. Before his death, however, he did provide a simple proof of the Erdös-Mordell inequality [2], and he gave a generalization of this proof to three dimensions. This generalization and his proof of (1) for trirectangular tetrahedra are given in Sections 2 and 3. We use this work as a basis for the proof of Theorem 1.

2. THE FUNDAMENTAL INEQUALITY

Let the vertices of S be i, j, k, and l; let (i) and (jkl) denote the area of the face opposite i, and (ij) the length of the edge joining i and j; let H_i be the length of the altitude through i, R the radius of the circumsphere with center O, R_i the distance from P to i, and r_i the distance from P to the face opposite i.

A theorem of Pappus plays a leading role in the proof of the Erdös-Mordell inequality given in [2]. The following generalization of this theorem to three dimensions is of importance in the proof of Theorem 1. *Construct three triangular prisms which have for their bases three faces of* S, *which have a lateral edge in common, and of which all or none lie entirely outside of* S. *On the remaining face, construct a fourth prism whose lateral edges are translates of the common lateral edge of the first three prisms. Then the sum of the volumes of the first three prisms is equal to the volume of the fourth prism.*

LEMMA 1. *For any tetrahedron* S,

$$\sum R_i \geq \sum \frac{(ij)^2 + (ik)^2 + (il)^2}{2RH_i} r_i . \tag{2}$$

Equality holds if and only if P *and* O *coincide.*

Proof. The ingredients of the proof are the generalized Pappus theorem, an inversion (a reflection was used in [2]), and a theorem of von Staudt [1, p. 117]. We

Received March 6, 1957.