On a Theorem of Frobenius

by

J. L. Ullman

1. Introduction. Let A be a matrix of order n with element a_{ij} in the i^{th} row and j^{th} column. The characteristic polynomial $P(\lambda)$ is the determinant $|A - \lambda I|$, where I is the identity matrix of order n. The roots of $P(\lambda)$ are called the eigenvalues of A. A theorem of Frobenius [1] states that if the a_{ij} are positive, then A has an eigenvalue which is positive, simple and exceeds the modulus of the other eigenvalues. Results about the eigenvalues of a matrix A for the case the elements a_{ij} are positive or zero can be deduced from this theorem by a limiting process. In section 2 a new, direct proof of a result for this case will be given. The extension of this result to the Fredholm integral equation will be discussed in section 3.

2. Statement and Proof of Theorem 1.

Definition 1. Let r be a positive integer. If r elements a_{ij} can be arranged to have the form

(1) $a_{t_1 t_2}, a_{t_2 t_3}, \ldots, a_{t_r t_1},$

they will be called a cycle of elements.

Theorem 1. Let A be a matrix of order n whose elements a_{ij} are positive or zero. The necessary and sufficient condition that A have a positive eigenvalue is that it have a cycle of elements, none of which vanish.