Close-to-Convex Schlicht Functions

by

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1. Principal results. Known theorems yield the following: if $\phi(z)$ is a convex schlicht function for $|z| < R$ and $f(z)$ is a function analytic for $|z| < R$ such that

$$\text{Re}\left[\frac{f'(z)}{\phi'(z)}\right] > 0, \quad |z| < R,$$

then $f(z)$ is also schlicht for $|z| < R$. Since the vectors f', ϕ' never differ in direction by more than 90°, it is natural to call f close-to-convex:

Definition. Let $f(z)$ be analytic for $|z| < R$. Then $f(z)$ is close-to-convex for $|z| < R$ if there exists a function $\phi(z)$, convex and schlicht for $|z| < R$, such that $f'(z)/\phi'(z)$ has positive real part for $|z| < R$.

When $R = 1$, it will be convenient to omit reference to the circular domain of definition. Therefore, a close-to-convex function will mean a function which is close-to-convex for $|z| < 1$.

We verify that the close-to-convex functions include several familiar classes of schlicht functions: e.g., the star functions, as well as some less familiar ones: e.g., the functions $f(z)$ having a Poisson integral representation in terms of a function $P(e^{i\theta})$ which is monotone in θ within each of two complementary arcs of $|z| = 1$.

It is of interest to characterize the close-to-convex functions intrinsically, without reference to a