Unusual Generating Functions for Ultraspherical Polynomials

by

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1. Introduction. A generating function equation for a set of polynomials $\{g_n(x)\}$ is an equation of type

$$(1) \qquad G(x, t) = \sum_{n=0}^{\infty} A_n \, g_n(x) \, t^n ,$$

where the coefficients A_n do not involve either x or t but may depend on the index of summation n, parameters (if any) of the set $\{g_n(x)\}$, and numerical factors. In many classical generating functions, the A_n is made up of ratios and products of factorial functions $(\alpha)_n$, where by definition

$$(2) \qquad (\alpha)_n = (\alpha+1) \ldots (\alpha+n-1) = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}$$

for $n = 1, 2, 3, \ldots$; $(\alpha)_0 = 1$ if $\alpha \neq 0$.

There has however been some interest paid to the case where the index of the factorial symbols making up A_n may be either n or $\left[\frac{n}{2}\right]$, the greatest integer in n/2. [1]

It is also obvious that some attention must be paid to the form permitted for $G(x, t)$. One of the most useful forms is that $G(x, t)$ consist of a finite sum of terms, each term of which is a product of a finite number of generalized hypergeometric functions,