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Multiplicative Functional for the Heat Equation on Manifolds with Boundary

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1. Introduction

By the Weitzenböck formula relating the Hodge–de Rham Laplacian and the covariant Laplacian for differential forms on a Riemannian manifold, the heat equation for differential forms is naturally associated with a matrix-valued Feynman–Kac multiplicative functional determined by the curvature tensor. The case of a closed manifold (without boundary) is well known and will be briefly reviewed below. In constrast, the case of manifolds with boundary is not well known, and for good reasons. Because the absolute boundary condition on differential forms is Dirichlet in the normal direction and Neumann in the tangential directions, the associated multiplicative functional is discontinuous and much more difficult to handle. Ikeda and Watanabe [6; 7] have dealt with this situation by using an excursion theory (for reflecting Brownian motion) that seems to have been created especially for this problem. In this paper we suggest a different approach that is based on an idea of approximation due to Airault [1]. This construction has the advantage that a key property of the multiplicative functional (i.e., the attendant Itô's formula for this functional) follows almost automatically from the approximate multiplicative functional without resorting to excursion theory, thus greatly simplifying this part of the theory; see Theorem 3.7.

Before coming to another and more important raison d'être for the present work, we briefly review some relevant facts for a closed manifold. Let M be a compact, closed Riemannian manifold and let α_0 be a 1-form on M. Consider the following initial value problem:

$$\begin{cases} \dfrac{\partial \alpha}{\partial t} = \dfrac{1}{2}\Box\alpha, \\ \alpha(\cdot, 0) = \alpha_0. \end{cases} \tag{1.1}$$

Here $\Box = -(d^*d + d^*d)$ is the Hodge–de Rham Laplacian on differential forms. Let $\Delta = \operatorname{trace}\nabla^2$ be the covariant Laplacian. Then we have the Weitzenböck formula

$$\Box\alpha = \Delta\alpha - \operatorname{Ric}\alpha,$$

where $\operatorname{Ric}_x : T_x^*M \to T_x^*M$ is the Ricci curvature transform. The solution can be represented probabilistically as follows. Let $\{x_t\}$ be a Brownian motion on M

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