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Chisini's Conjecture for Curves with Singularities of Type $x^n = y^m$

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1. Introduction

This paper is devoted to a classical problem that can be summarized as follows: Let S be a nonsingular compact complex surface, let $\pi: S \to \mathbb{P}^2$ be a finite morphism having simple branching, and let B be the branch curve; then (cf. [F2]), "to what extent does B determine $\pi: S \to \mathbb{P}^2$"?

The problem was first studied by Chisini [Ch], who proved that B determines S and π, assuming (i) B to have only nodes and cusps as singularities, (ii) the degree d of π to be greater than 5, and (iii) a strong hypothesis on the possible degenerations of B. Chisini posed the question of whether the first or the third hypothesis could be weakened. More recently, Kulikov [Ku] and Nemirovski [Ne] proved the result for $d \geq 12$, assuming B to have only nodes and cusps as singularities.

In this paper we weaken the hypothesis about the singularities of B: we generalize the theorem of Kulikov and Nemirovski for B having only singularities of type $\{x^n = y^m\}$, using the additional hypothesis of smoothness for the ramification divisor (automatic in the "nodes and cusps" case). Moreover, we exhibit a family of counterexamples showing that our additional hypothesis is necessary.

In order to more precisely state the problem and our results, we need to introduce a bit of notation.

Definition 1.1. A *normal generic cover* is a finite holomorphic map $\pi: S \to \mathbb{C}^2$, which is an analytic cover branched over a curve B such that S is a connected normal surface and the fiber over a smooth point of B is supported on $\deg \pi - 1$ distinct points.

Two normal generic covers (S_1, π_1), (S_2, π_2) with the same branch locus B are called (analytically) *equivalent* if there exists an isomorphism $\phi: S_1 \to S_2$ such that $\pi_1 = \pi_2 \circ \phi$.

The main interest in generic covers comes from the well-known fact that, by the Weierstrass preparation theorem, given an analytic surface $S \subset \mathbb{C}^n$, a generic

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