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Some Applications of Bruhat–Tits Theory to Harmonic Analysis on a Reductive p-adic Group

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1. Introduction

Let k denote a field with nontrivial discrete valuation. We assume that k is complete with perfect residue field $\mathfrak{f}$. Let G denote the group of k-rational points of a reductive, connected, linear algebraic group $\mathbf{G}$ defined over k and let $\mathfrak{g}$ denote its Lie algebra. Let $\mathcal{B}(G)$ denote the Bruhat–Tits building of G.

The basic tools of harmonic analysis on $\mathfrak{g}$ are invariant distributions and the Fourier transform. In [1; 12] the formalism of Moy and Prasad [17; 18] is used to develop a "uniform" way to describe both the support of invariant distributions and how certain important spaces of functions behave with respect to the Fourier transform. The purpose of this paper is to prove the group analogues of these results. As discussed below, these results are more difficult to obtain than their Lie algebra counterparts.

We begin by studying a relationship between the structure of G and the geometry associated to the displacement function on $\mathcal{B}(G)$. Fix $g \in G$. In Section 3.1 we associate to g a Levi subgroup, M_g. We then show that either g fixes a point in $\mathcal{B}(G)$ or there is a line in $\mathcal{B}(G)$ on which g acts by nontrivial translation, but not both. (A line in a building is a 1-dimensional affine subspace of an apartment.) This result (Corollary 3.1.5) uses the nonpositive curvature of $\mathcal{B}(G)$, and it is the basis for many of the results of the paper. Define the *displacement function* d_g on $\mathcal{B}(G)$ by setting $\mathrm{d}_g(x)$ equal to the distance g moves x. We show that the set of elements in $\mathcal{B}(G)$ where d_g assumes its minimum value is nonempty. It follows [7, Chap. II] that the subset of $\mathcal{B}(G)$ where d_g assumes its minimum value can be characterized as either the set of g-fixed points in $\mathcal{B}(G)$ or the union of lines in $\mathcal{B}(G)$ on which g acts by nontrivial translation. We then show that if ℓ is a line on which g acts by nontrivial translation, then the Levi subgroup M_g is equal to the Levi subgroup of G naturally associated to ℓ.

Suppose $r \geq 0$. We next obtain group analogues of the results on $\mathfrak{g}$ of [12, Sec. 1.6] (see also [1, Sec. 3.1]); these results are used to describe the support of invariant distributions. We show that

$$\bigcup_{x \in \mathcal{B}(G)} G_{x,r} = \bigcap_{x \in \mathcal{B}(G)} G_{x,r} \cdot \mathcal{U}. \qquad (\dagger)$$

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