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Affine Surfaces with $AK(S) = \mathbb{C}$

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1. Introduction

In this paper we proceed with our research [BaM1; BaM2] of the smooth surfaces with $\mathbb{C}^+$-actions. We denote by $\mathcal{O}(S)$ the ring of all regular functions on S. Let us recall that the AK invariant $AK(S) \subset \mathcal{O}(S)$ of a surface S is just the subring of the ring $\mathcal{O}(S)$ consisting of those regular functions on S that are invariant under all $\mathbb{C}^+$-actions of S. This invariant can be also described as the subring of $\mathcal{O}(S)$ of all functions that are constants for all locally nilpotent derivations of $\mathcal{O}(S)$ [KKMR; KM; M1].

We would like to give the answer to the following question: What are the surfaces with the trivial invariant AK?

It is quite easy to show (see [M2]) that the complex line $\mathbb{C}$ is the only curve with the trivial invariant. It is also well known that, if $AK(S) = \mathbb{C}$ and $\mathcal{O}(S)$ is a unique factorization domain (UFD), then S is an affine complex plane $\mathbb{C}^2$ [MiS; S]. If we drop the UFD condition then we have many smooth surfaces with trivial invariant—for example, any hypersurface of the form $\{xy = p(z)\} \subset \mathbb{C}^3$, where all roots of $p(z)$ are simple.

Since we did not know any other examples, we had the following working conjecture.

Conjecture. *Any smooth affine surface S with $AK(S) = \mathbb{C}$ is isomorphic to a hypersurface*

$$\{xy = p(z)\} \subset \mathbb{C}^3.$$

It turned out that this conjecture is true only with an additional assumption that S admits a fixed-point–free $\mathbb{C}^+$-action. Also, if we assume that S is a hypersurface with $AK(S) = \mathbb{C}$ then S is indeed isomorphic to a hypersurface defined by the equation $xy = p(z)$.

Surfaces of this kind have been well known since 1989 owing to the following remarkable fact, which was discovered by Danielewski [D] in connection with the generalized Zariski conjecture (see also Fieseler [F]): the surfaces $\{x^n y = p(z)\}$

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